108. An Approach by Difference to a Quasi-Linear Parabolic Equation

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1. Introduction. This paper treats the semi-group associated with the Cauchy problem for the equation

(1) $\partial u/\partial t=\Delta\phi(u)$ for $t>0$ and $x\in R^N$ $\left(\Delta=\sum_{i=1}^{N}\partial^2/\partial x_i^2\right)$

through the difference scheme

(2) $$h^{-1}(u(t+h,x)-u)=\sum_{i=1}^{N}L^{-2}\{\phi(u(t,x+Le_i))-2\phi(u)+\phi(u(t,x-Le_i))\},\quad (e_i=(0,\cdots,0,\overset{i}{1},0,\cdots,0))$$

where ϕ is a differentiable function on R with $\phi(0)=0$ such that ϕ' is non-negative and bounded on every finite sub-interval of R. The convention

(3) $$C_i(t)f(x)=(f(x+te_i)+f(x-te_i))/2\qquad(i=1,\cdots,N)$$

enables us to rewrite (2) as

(2)′ $$h^{-1}(u(t+h,x)-u(t,x))=\sum_{i=1}^{N}2L^{-2}(C_i(L)-I)\phi(u(t,x)),$$

and provides a strongly continuous cosine family $C_i(t)$, $t\in R$ in a Banach space $L^1(R^N)$ with norm $\|\cdot\|_1$ for each fixed i. For cosine families in Banach spaces, see [7] for example.

The Cauchy problem for (1) arises in mathematical models of many physical situations. The semi-group approaches to (1) in $L^1(R^N)$ were made by Benilan, Brezis and Crandall (see [2], [4]). The method is essentially based on their theory on the semi-linear equation $\phi^{-1}(u)-\Delta u=f$ developed in [1]. But, our method is more constructive and provides applications to numerical analysis for (1). Indeed, our main task is to show that

$$\left(I-\lambda\sum_{i=1}^{N}2L^{-2}(C_i(L)-I)\phi\right)^{-1}\quad\text{converges in } L^1(R^N) \text{ as } L\downarrow 0.$$

2. Main results. Consider the operator C_h defined by

(4) $$C_hu=u+h\sum_{i=1}^{N}2L^{-2}(C_i(L)-I)\phi(u),$$

where $h, L>0$ and $L^2=2Nh\sup_{|r|\leq m}\phi'(r)$ for an integer m. Let A_i be, for each i, the infinitesimal generator of the strongly continuous cosine family $C_i(t)$, $t\in R$ in $L^1(R^N)$ defined by (3), and let $\overline{A}$ be the smallest closed extension of $\sum_{i=1}^{N}A_i$ in $L^1(R^N)$. We are concerned with a generalization A_ϕ in $L^1(R^N)$ of $\Delta\phi$ defined by

$$\begin{cases} A_\phi u = \overline{A}\phi(u) & \text{for } u \in D(A_\phi), \\ D(A_\phi) = \{u \in L^1(R^N) \cap L^\infty(R^N) : \phi(u) \in D(\overline{A})\}. \end{cases}$$

Theorem. *A_ϕ is a dissipative operator with domain $D(A_\phi)$ dense in $L^1(R^N)$ satisfying the range condition:*

$$(5) \qquad R(I-\lambda A_\phi) \supset L^1(R^N) \cap L^\infty(R^N) \qquad \text{for } \lambda > 0.$$

For every $\lambda>0$ and $u \in L^1(R^N) \cap L^\infty(R^N)$ with $\|u\|_\infty \leqq m$

$$(6) \qquad (I-\lambda h^{-1}(C_h - I))^{-1}u \longrightarrow (I-\lambda A_\phi)^{-1}u \quad \text{in } L^1(R^N) \quad \text{as } h \downarrow 0.$$

Let $S_\phi(t)$, $t>0$ be the contraction semi-group in $L^1(R^N)$ generated by A_ϕ in the sense of Crandall-Liggett. Then, Brezis-Pazy's convergence theorem [5, Theorem 3.2] is applicable to yield the result that for $u \in L^1(R^N) \cap L^\infty(R^N)$ with $\|u\|_\infty \leqq m$

$$(7) \qquad C_h^{[t/h]}u \longrightarrow S_\phi(t)u \quad \text{in } L^1(R^N) \quad \text{as } h \downarrow 0$$

uniformly on every finite sub-interval of $[0, \infty)$. This formula enables us to know some other properties of $S_\phi(t)$, $t>0$ by means of C_h. A similar result to (7) was obtained by [3] with C_h otherwise defined.

3. Lemmas. Let X_m be, for an integer m, the totality of $u \in L^1(R^N) \cap L^\infty(R^N)$ such that $\|u\|_\infty \leqq m$.

Lemma 1. *For each $h>0$, C_h maps X_m into itself and satisfies for every $u, v \in X_m$*

(i) $\|C_h u - C_h v\|_1 \leqq \|u-v\|_1$, (ii) $\|C_h u\|_\infty \leqq \|u\|_\infty$.

Proof. Since $r-2NhL^{-2}\phi(r)$ is non-decreasing in $r \in [-m, m]$ and $|r-s-2NhL^{-2}(\phi(r)-\phi(s))|+2NhL^{-2}|\phi(r)-\phi(s)|=|r-s|$ for $r, s \in [-m, m]$, the lemma is clear from (4). Q.E.D.

Lemma 2. *For each $h>0$, $A_h = h^{-1}(C_h - I)$ satisfies*

(i) $\operatorname{sgn}(u) \cdot A_h u \leqq \sum_{i=1}^N 2L^{-2}(C_i(L)-I)|\phi(u)|$,

(ii) $\int_{R^N} \operatorname{sgn}(u) \cdot A_h u f(x)dx \leqq \sup_{|r| \leqq m} \phi'(r) \cdot \|u\|_1 \|\Delta f\|_\infty$

for every $u \in X_m$, where f is an arbitrary function on R^N such that $f(x) \geqq 0$ for $x \in R^N$ and $\Delta f \in L^\infty(R^N)$.

Proof. Since $A_h = \sum_{i=1}^N 2L^{-2}(C_i(L)-I)\phi(\cdot)$, (i) is clear. Multiplication by $f(x)$ and integration of (i) yield (ii). Q.E.D.

Lemma 3. *$\overline{A}$ is the infinitesimal generator of a linear contraction semi-group $T(t)$, $t>0$ in $L^1(R^N)$ represented by*

$$(8) \qquad T(t)u(x) = (4\pi t)^{-N/2} \int_{R^N} e^{-|x-y|^2/(4t)} u(y)dy \quad \left(|x|^2 = \sum_{i=1}^N x_i^2\right)$$

and satisfies for every $u \in L^1(R^N)$

$$(9) \qquad \left(I - \lambda \sum_{i=1}^N 2t^{-2}(C_i(t)-I)\right)^{-1} u \longrightarrow (I-\lambda\overline{A})^{-1}u \quad \text{in } L^1(R^N) \quad \text{as } t \downarrow 0.$$

Proof. $T(t)$, $t>0$ is given by $T(t) = T_1(t) \cdots T_N(t)$ with

$$T_i(t)u = (\pi t)^{-1/2} \int_0^\infty e^{-s^2/(4t)} C_i(s)u \, ds, \quad u \in L^1(R^N), \quad (i=1, \cdots, N)$$

which was suggested by Fattorini [6, p. 92]. Since $C_i(t)$ and $C_j(s)$

commute for any t, $s \in R$ and $i, j=1, \cdots, N$, the proof of the lemma is simple and standard. Q.E.D.

From (8) we see that for every $t>0$ and $u, v \in X_m$

$$\int_{R^N} \operatorname{sgn}(u-v)\cdot t^{-1}(T(t)-I)(\phi(u)-\phi(v))dx \leqq 0. \tag{10}$$

4. Proof of Theorem. Lemma 1 implies that for each $h>0$, $J_\lambda^h=(I-\lambda A_h)^{-1}$ exists and satisfies for every $\lambda>0$ and a given $u \in X_m$

$$\|J_\lambda^h u\|_1 \leqq \|u\|_1, \tag{11}$$

$$\|J_\lambda^h u(\cdot+k)-J_\lambda^h u\|_1 \leqq \|u(\cdot+k)-u\|_1, \qquad k \in R^N. \tag{12}$$

Replacing $f(x)$ by $\psi(2|x|/\rho-1)(\rho>0)$ and u by $J_\lambda^h u$ in Lemma 2, (ii), we obtain

$$\begin{aligned}(13)\quad &\int_{|x|>\rho} |J_\lambda^h u|\, dx \\ &\leqq \int_{|x|>\rho/2} |u|\, dx + 4\lambda\rho^{-2} \sup_{|r|\leqq m} \phi'(r)\cdot\|u\|_1(\|\psi''\|_\infty+(N-1)\|\psi'\|_\infty),\end{aligned}$$

where ψ is a very smooth function: $R\to[0,1]$ with values 0 for $r\leqq 0$ and 1 for $r\geqq 1$.

Let $\{h_n\}$ be a sequence vanishing as $n\to\infty$. Then, Lemmas 1 and 2 hold for all C_{h_n} defined by (4) with $h=h_n$, $n=1, 2, \cdots$. Noting (11)–(13) for $J_\lambda^{h_n}$ and using the Fréchet-Kolmogorov theorem, we can prove that there is a sub-sequence denoted again by $\{h_n\}$ such that $J_\lambda^{h_n}u$ converges to some u_λ in $L^1(R^N)$. But, the equality

$$\begin{aligned}&\left(I-\mu\sum_{i=1}^N 2L^{-2}(C_i(L)-I)\right)^{-1}\lambda^{-1}(J_\lambda^h u-u) \\ &\quad =\mu^{-1}\left\{\left(I-\mu\sum_{i=1}^N 2L^{-2}(C_i(L)-I)\right)^{-1}-I\right\}\phi(J_\lambda^h u)\end{aligned}$$

together with (9) implies

$$(I-\mu\overline{A})^{-1}\lambda^{-1}(u_\lambda-u)=\mu^{-1}((I-\mu\overline{A})^{-1}-I)\phi(u_\lambda), \qquad \mu>0$$

$$\text{i.e.}\quad u_\lambda-\lambda A_\phi u_\lambda=u \quad \text{with} \quad u_\lambda \in D(A_\phi).$$

The dissipativeness of A_ϕ follows from (10). The denseness of $D(A_\phi)$ can be proved by showing that for every $u \in X_m$ there exists a sequence $\{\lambda_n\}$ vanishing as $n\to\infty$ such that $(I-\lambda_n A_\phi)^{-1}u$ converges to u in $L^1(R^N)$. We have only to derive (11)–(13) with $J_\lambda^h u$ replaced by $(I-\lambda A_\phi)^{-1}u$ and to use the Fréchet-Kolmogorov theorem again.

References

[1] Ph. Benilan, H. Brezis, and M. G. Crandall: A semilinear elliptic equation in $L^1(R^N)$. Ann. Scuola Norm. Pisa, **2**, 523–555 (1975).

[2] Ph. Benilan and M. G. Crandall: The continuous dependence on ϕ of solutions of $u_t-\Delta\phi(u)=0$. MRC Technical Summary Report (1979).

[3] A. E. Berger, H. Brezis, and C. W. Rogers: A numerical method for solving the problem $u_t-\Delta f(u)=0$. R.A.I.R.O. Analyse numerique/Numerical Analysis, **13**, 297–312 (1979).

[4] H. Brezis and M. G. Crandall: Uniqueness of solutions of the initial value problem for $u_t - \Delta\phi(u) = 0$. J. Math. Pures Appl., **58**, 153–163 (1979).

[5] H. Brezis and A. Pazy: Convergence and approximation of semigroups of nonlinear operators in Banach spaces. J. Func. Anal., **9**, 63–74 (1972).

[6] H. O. Fattorini: Ordinary differential equations in linear topological spaces, I. J. Differential Equations, **5**, 72–105 (1968).

[7] C. Travis and G. Webb: Second order differential equations in Banach space. Nonlinear Equations in Abstract Spaces (ed. by V. Lakshmikantham). pp. 331–361 (1978).