48. On the Resolution of Two-dimensional Singularities

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§1. **Introduction.** Let $f(z_1, \cdots, z_n)$ be a germ of an analytic function at the origin such that $f(0)=0$ and f has an isolated critical point at the origin. We assume that f has a non-degenerate Newton boundary. Let V be a germ of hypersurface $f^{-1}(0)$. Let $\Gamma^*(f)$ be the dual Newton diagram and let Σ^* be a simplicial subdivision of $\Gamma^*(f)$. It is known that there is a canonical resolution $\pi: \tilde{V} \to V$ which is associated with Σ^*. ([1]). However the process to get Σ^* from $\Gamma^*(f)$ is not unique and a "bad" Σ^* gives unnecessary exceptional divisors. The purpose of this paper is to show that in the case $n=3$, there is a canonical subdivision Σ^* of $\Gamma^*(f)$ so that the resolution graph is obtained by a canonical surgery from $S_2\Gamma^*(f)$ (=two-skeleton of $\Gamma^*(f)$). See Theorem (5.1).

§2. **Newton boundary and the dual Newton diagram.** Let $f(z_1, \cdots, z_n)=\sum_\nu a_\nu z^\nu$ be the Taylor expansion of f where $z^\nu = z_1^{\nu_1}\cdots z_n^{\nu_n}$. Recall that the Newton boundary $\Gamma(f)$ is the union of the compact faces of $\Gamma_+(f)$ where $\Gamma_+(f)$ is the convex hull of the union of the subsets $\{\nu+(\boldsymbol{R}^+)^n\}$ for ν such that $a_\nu \neq 0$. For any closed face Δ of $\Gamma(f)$, we associate the polynomial $f_\Delta(z)=\sum_{\nu\in\Delta} a_\nu z^\nu$. We say that f is *non-degenerate* if f_Δ has no critical point in $(\boldsymbol{C}^*)^n$ for any $\Delta \in \Gamma(f)$ ([2]).

Let N^+ be the space of positive vectors in the dual space of $\boldsymbol{R}^n$. For any vector $P={}^t(p_1, \cdots, p_n)$ of N^+, we associate the linear function $P(x)=\sum_i p_i x_i$ on $\Gamma_+(f)$ and let $d(P)$ be the minimal value of $P(x)$ on $\Gamma_+(f)$ and let $\Delta(P)=\{x\in\Gamma_+(f)\,;\,P(x)=d(P)\}$. We introduce an equivalence relation $\sim$ on N^+ by $P\sim Q$ if and only if $\Delta(P)=\Delta(Q)$. For any face Δ of $\Gamma_+(f)$, let $\Delta^*=\{P\in N^+\,;\,\Delta(P)=\Delta\}$. The collection of Δ^* gives a polyhedral decomposition of N^+ which we call *the dual Newton diagram of f* and we denote it by $\Gamma^*(f)$. $\Delta(P)$ is a compact face of $\Gamma(f)$ if and only if P is strictly positive. We say that a subdivision Σ^* of $\Gamma^*(f)$ is a *simplicial subdivision* if the following conditions are satisfied ([1]).

(i) Σ^* is a subdivision by the cones over a simplicial polyhedron whose simplexes are spanned by primitive integral vectors with determinant ± 1 in the sense of §3.

(ii) Let $\sigma=(P_1, \cdots, P_n)$ be an $(n-1)$-simplex. Then there exists a permutation τ of $\{1, \cdots, n\}$ such that

(2.1) $$\Delta(P_{\tau(1)}) \supset \Delta(P_{\tau(2)}) \supset \cdots \supset \Delta(P_{\tau(n)}).$$

§3. Canonical simplicial subdivision. Let $P_i = {}^t(p_{1,i}, \cdots, p_{n,i})$ $(i=1, \cdots, k)$ be given primitive integral vectors of N^+. We define a non-negative integer $\det(P_1, \cdots, P_k)$ by the greatest common divisor of all $k \times k$ minors of the matrix $(p_{j,i})$ and we call $\det(p_1, \cdots, p_k)$ *the determinant of* $P_1, \cdots, P_k$.

Lemma (3.1). *Let P and Q be given primitive integral vectors in N^+. Let $c = \det(P, Q)$ and assume that $c > 1$. There exists a unique integer c_1 such that $0 < c_1 < c$ and $P_1 = (Q + c_1 P)/c$ is an integral vector on $\overline{PQ}$. We have* $\det(P, P_1) = 1$ *and* $\det(P_1, Q) = c_1$.

Remark. By the abuse of language, we say that P_1 is on $\overline{PQ}$ if the normalized vector $P_1' = P_1/a$ is on $\overline{PQ}$ where $a = (1 + c_1)/c$.

Definition. Let $\overline{PQ}$ be a line segment of $S_2\Gamma^*(f)$. We say that primitive vectors $\{P_1, \cdots, P_k\}$ is *the canonical primitive sequence* on $\overline{PQ}$ if the followings are satisfied.

(i) Let $c = \det(P, Q)$ and assume that $c > 1$. There exists positive integers $c = c_0 > c_1 > \cdots > c_k = 1$ such that $P_{i+1} = (Q + c_{i+1} P_i)/c_i$ for each i. $(P_0 = P.)$

(ii) If $c = 1$, $k = 1$ and $P_1 = P + Q$.

Lemma (3.2). *Assume that* $c = \det(P, Q) > 1$. *Let $P_1, \cdots, P_k$ be the canonical primitive sequence on $\overline{PQ}$ and let c_i $(i=1, \cdots, k)$ be as above. Let $m_i = (c_{i-1} + c_{i+1})/c_i$. $(c_{k+1} = 0.)$ Then m_i $(i=1, \cdots, k)$ are integers and $m_i \geqq 2$ and the continuous fraction*

$$m_1 - \cfrac{1}{m_2 - \cdots - \cfrac{1}{m_k}}$$

is equal to c/c_1. Let $P_i = {}^t(p_{1,i}, \cdots, p_{n,i})$. Then $m_i = (p_{j,i-1} + p_{j,i+1})/p_{j,i}$ for each j.

We say that a simplicial subdivision Σ^* is *canonical* if it gives the canonical primitive sequence on each line segment $\overline{PQ}$ of $S_2\Gamma^*(f)$. The existence is derived from the following lemma $(n=3)$.

Lemma (3.3). *Let Δ be a triangle with primitive vectors P, Q and R as vertices. Let* $c = \det(P, Q, R)$. *We assume that* $\det(p, Q) = \det(P, R) = 1$ *and $c > 1$. Then there exist unique c_1 and d_1 such that $0 < c_1 < c$, $0 \leqq d_1 < c$ and $T_1 = (R + c_1 Q + d_1 P)/c$ is an integral vector. T_1 divides Δ into three triangles with* $\det(P, Q, T_1) = 1$, $\det(P, T_1, R) = c_1$, $\det(Q, T_1, R) = d_1$.

§4. Resolution of V. Let Σ^* be a simplicial subdivision of $\Gamma^*(f)$. For each $(n-1)$-simplex $\sigma = (P_1, \cdots, P_n)$, we associate an n-dimensional Euclidean space $\boldsymbol{C}^n$ with coordinates $(y_{\sigma,1}, \cdots, y_{\sigma,n})$ and a birational mapping $\pi_\sigma: \boldsymbol{C}_\sigma^n \to \boldsymbol{C}_\sigma^n$ which is defined by $z_i = y_{\sigma,1}^{p_{i,1}} \cdots y_{\sigma,n}^{p_{i,n}}$. Let X be the union of $\boldsymbol{C}_\sigma^n$ which are glued along the images of π_σ. Let π be the projection and let $\tilde{V}$ be the closure of $\pi^{-1}(V \cap (\boldsymbol{C}^*)^n)$. It is known that $\pi: \tilde{V} \to V$ is a resolution of V ([1]). Let $d_i = d(P_i)$ and $\Delta_i = \Delta(P_i)$. We assume that $\Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_n$. We define $f_\sigma(\boldsymbol{y}_\sigma)$ and $g_{\Delta_i}(\boldsymbol{y}_\sigma)$ by $f(\pi_\sigma(\boldsymbol{y}_\sigma)) = f_\sigma(\boldsymbol{y}_\sigma) \prod_i y_{\sigma,i}^{d_i}$ and $f_{\Delta_i}(\pi_\sigma(\boldsymbol{y}_\sigma)) = g_{\Delta_i}(\boldsymbol{y}_\sigma) \prod_i y_{\sigma,i}^{d_i}$. By the definition, $\tilde{V}$ is defined by $f_\sigma(\boldsymbol{y}_\sigma) = 0$ and $\tilde{V} \cap \{y_{\sigma,i} = 0\}$ is $\{\boldsymbol{y}_\sigma ; y_{\sigma,i} = 0$ and $g_{\Delta_i}(\boldsymbol{y}_\sigma) = 0\}$. Note that $g_{\Delta_i}(\boldsymbol{y}_\sigma)$ is a function of $y_{\sigma,i+1}, \cdots, y_{\sigma,n}$. Thus $\tilde{V} \cap \{y_{\sigma,i} = 0\}$ is non-empty if and only if $\dim \Delta_i > 0$. Let $E(P_i ; \sigma) = \{\boldsymbol{y}_\sigma \in \tilde{V} ; y_{\sigma,i} = 0\}$. $\pi(E(P_i ; \sigma)) = \{0\}$ if and only if P_i is strictly positive. The union of $E(P_i ; \sigma)$ for simplexes σ which contain P_i is a divisor of V and we denote it by $E(P_i)$. We say that vertices $P_1, \cdots, P_k$ in Σ^* are *adjacent* if there is an $(n-1)$-simplex σ of Σ^* which contains $P_1, \cdots, P_k$.

Lemma (4.1). *Let $P_1, \cdots, P_k$ be vertices of Σ^* with* $\dim \Delta(P_i) \geqq 1$. *$\bigcap_i E(P_i)$ is non-empty if and only if $P_1, \cdots, P_k$ are adjacent.*

Lemma (4.2). *Assume that P is a strictly positive vertex of Σ^* such that* $\dim \Delta(P) = 1$. *Then $E(P)$ has $r(P)+1$ connected components. If $n=3$, they are rational curves. Here $r(P)$ is the number of the integral points in $\Delta(P) - \partial\Delta(P)$.*

Let $g(u_1, \cdots, u_k)$ be a polynomial with support $S(g)$. We say that g is *globally non-degenerate* (=0-*non-degenerate* in [7]) if $g_\Delta(u)$ has no critical point in $(\boldsymbol{C}^*)^k \cap g_\Delta^{-1}(0)$ for each Δ.

The exceptional divisor $E(P)$ has a canonical stratification in which each stratum is described by $g^{-1}(0)$ for some globally non-degenerate polynomial g.

Lemma (4.3) ([2], [5], [7]). *Let $g(u_1, \cdots, u_k)$ be a globally non-degenerate polynomial and let $V^* = g^{-1}(0) \cap (\boldsymbol{C}^*)^k$. Then the Euler characteristic of V^* is $(-1)^{k+1} k!$* k-dim. volume $S(g)$.

§ 5. Main result. We assume that $n=3$ and let $\pi: \tilde{V} \to V$ be the good resolution associated with Σ^*. Let Δ be a two dimensional face of $\Gamma(f)$. We define $g(\Delta)$ by the number of the integral points in $\Delta - \partial\Delta$. Our main result is

Theorem (5.1). *Let $\pi: \tilde{V} \to V$ be as above. Then for a strict-positive vertex of Σ^*, we have*

(i) *If* $\dim \Delta(P) = 2$, *$E(P)$ has genus $g(\Delta(P))$.*

(ii) *If* $\dim \Delta(P) = 1$, *$E(P)$ is a disjoint union of $r(P)+1$ rational curves.*

(iii) *Assume that Σ^* is canonical. Then the resolution graph is obtained by a canonical surgery of $\Gamma^*(f)$ as follows: Let $\overline{PQ}$ be a line segment of $\Gamma^*(f)$ and assume that P is strictly positive. Let* $c = \det(P, Q)$ *and assume that $c > 1$. Let c_1 be as Lemma* (3.1). *Let*

$$m_1 = \cfrac{1}{m_2 - \ddots - \cfrac{1}{m_k}}$$

be the continuous fraction of c/c_1. *We insert* $r(P, Q)+1$ *copies of chains of rational curves* $\overset{-m_1}{-\!\!-\!\!\bullet}\!\!-\!\!-\!\!\overset{-m_2}{\bullet}\!\!-\!\cdots\!-\!\!\overset{-m_k}{\bullet}\!\!-\!\!-$ *between* P *and* Q. *Here* $r(P, Q)=r(P+Q)$. *In the case of* $c=1$, *the chain is* $-\!\!-\!\!\overset{-1}{\bullet}\!\!-\!\!-$ *by definition. If neither* P *nor* Q *is strictly positive, we do nothing. Those vertices which are not strictly positive are omitted from the resolution diagram after the surgery. Assume that* $\dim \Delta(P)=2$. *Let* $Q_1, \cdots, Q_s$ *be the vertices of* Σ^* *which are adjacent to* P. *Let* $P={}^t(p_1, p_2, p_3)$ *and* $Q_i={}^t(q_{1,i}, q_{2,i}, q_{3,i})$ $(i=1, \cdots, s)$. (s *is the number of one-dimensional boundaries of* $\Delta(P)$.) *Then the self-intersection number of* $E(P)$ *is* $-\sum_i^s (r(P, Q_i)+1)q_{1,i})/p_1$.

The proof is done by considering the divisor of the holomorphic function $\pi^* z_1$ on $\tilde{V}$ and by the property $(\pi^* z_1)\cdot E(P)=0$. Lemmas (3.2) and (4.3) and the following lemma play the key role in the proof.

Lemma (5.1). *Let* Δ *be a compact polyhedron in* $\boldsymbol{R}^2$ *with integral points as vertices. Let* $\Delta_1, \cdots, \Delta_s$ *be one dimensional faces of* Δ. *Then we have* $2 \text{ volume } \Delta = 2g(\Delta)+\sum_i^s (r(\Delta_i)+1)-2$.

Further details will be treated in [6].

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