9. *Propagation of Singularities of Solutions to Semilinear Schrödinger Equations*

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The purpose of this note is to study micro-local singularities of solutions to some semilinear Schrödinger equation. In [4], Rauch studied singularities of classical solutions to the equation $\square u=f(u)$ and showed that singularities modulo H^r, for some r, propagate along null bicharacteristic strips. Here, we follow his arguments and obtain a similar result for semilinear Schrödinger equations.

1. Notation and statement of the result. Let Ω denote an open set of $\boldsymbol{R}^n$. Let $M=(\mu_1, \cdots, \mu_n)$ be a multiweight on the dual space $\boldsymbol{R}_n$, with $\inf\{\mu_j\}=1$. If $\xi \in \boldsymbol{R}_n$ and $t>0$ we shall use the notation $t^M\xi=(t^{\mu_1}\xi_1, \cdots, t^{\mu_n}\xi_n)$. We shall say that a function g on $\Omega\times(\boldsymbol{R}_n\backslash 0)$ is (M-) quasi-homogeneous of degree m if $g(x, t^M\xi)=t^m g(x, \xi)$ for $t>0$, and that a subset Γ of $\Omega\times(\boldsymbol{R}_n\backslash 0)$ is a M-cone if $(x, \xi)\in\Gamma$ implies $(x, t^M\xi)\in\Gamma$ for every $t>0$. We introduce the function $[\cdot]_M$ defined implicitly by $\sum \xi_j^2/[\xi]_M^{2\mu_j}=1$ if $\xi\neq 0$ and $[0]_M=0$.

We let $S_M^m(\Omega)$ denote the space of C^∞-functions $p: \Omega\times\boldsymbol{R}_n\to\boldsymbol{C}$ satisfying the following estimate: for every $\alpha, \beta\in \boldsymbol{N}^n$, $K\subset\subset\Omega$ there exists a constant $C=C_{\alpha\beta K}$ such that

$$|\partial_\xi^\alpha\partial_x^\beta p(x, \xi)|\leq C(1+[\xi]_M)^{m-\langle\alpha, M\rangle} \qquad \text{for } x\in K,$$

where $\langle\alpha, M\rangle=\sum\alpha_j\mu_j$. If $p\in S_M^m(\Omega)$ we set

$$p(x, D_x)u(x)=(2\pi)^{-n}\iint e^{i\langle x-y, \xi\rangle}p(x, \xi)u(y)dyd\xi \qquad \text{for } u\in C_0^\infty(\Omega),$$

and use the terminology of *M-pseudo-differential operators* for it. We shall say that $p\in S_M^m(\Omega)$ is a classical symbol if p has an asymptotic expansion by quasi-homogeneous functions p_{m_j} of degree m_j: $p(x, \xi)\sim p_m(x, \xi)+\sum_{j=1}^\infty p_{m_j}(x, \xi)$, with $m-1\geq m_1>m_2>\cdots$. For a classical symbol $p\in S_M^m(\Omega)$ we call the top term p_m principal symbol and define its M-Hamiltonian vector field in $\Omega\times(\boldsymbol{R}_n\backslash 0)$ to be $\sum_{\mu_j=1}(\partial_{\xi_j}p_m\partial_{x_j}-\partial_{x_j}p_m\partial_{\xi_j})$ which is denoted by H_p^M. To the classical M-pseudo-differential operator with real principal symbol, a bicharacteristic strip is an integral curve of the M-Hamiltonian vector field.

Let $H_M^s(\Omega)$ be a weighted Sobolev space with the norm

$$\|u\|_{M,s}=\|(1+[\xi]_M)^s\hat{u}(\xi)\|_{L^2} \qquad \text{for } u\in C_0^\infty(\Omega).$$

We also define its micro-localization:

Definition. Let $u\in\mathcal{D}'(\Omega)$ and $z_0\in\Omega\times(\boldsymbol{R}_n\backslash 0)$. The implication $u\in H_M^s(z_0)$ means that there exists a classical symbol $a(x, \xi)\in S_M^0(\Omega)$ such that $a_0(z_0)\neq 0$ and $a(x, D_x)u\in H_M^s(\Omega)$. (We then say that u belongs to H_M^s at z_0.)

Furthermore, let $\Gamma\subset\Omega\times(\boldsymbol{R}_n\backslash 0)$ be an M-cone. Then we write $u\in H^s_M(\Gamma)$ if u belongs to H^s_M at all points of Γ. And as usual, $H_{M,\mathrm{loc}}(\Omega)$ denotes the space of all functions which belong to H^s_M at every point of $\Omega\times(\boldsymbol{R}_n\backslash 0)$.

Let $-i\partial_t-\Delta$ be the Schrödinger operator in $\boldsymbol{R}_t\times\boldsymbol{R}^n_x$. Then its symbol $p=\tau+|\xi|^2$ is real and by taking $M=(2,1,\cdots,1)$ this belongs to $S^2_M(\boldsymbol{R}_t\times\boldsymbol{R}^n_x)$. Since the Hamiltonian vector field is $2\sum\xi_j\partial_{x_j}$ we see that a bicharacteristic strip of Schrödinger operator is a straight line in the hyperplane $t=$constant. Now, let $f(u,\bar{u})$ be a holomorphic function of two complex variables and Ω be an open subset of $\boldsymbol{R}_t\times\boldsymbol{R}^n_x$. We consider the semilinear equation:

$$-i\partial_t u-\Delta u=f(u,\bar{u})\qquad\text{in }\Omega. \tag{1.1}$$

Our result is

Theorem. *Let u be a solution of* (1.1) *belonging to $H^s_{M,\mathrm{loc}}(\Omega)$ for $s>(n+2)/2$ and let $\sigma\leq s-(n+2)/2$. If $u\in H^{s+\sigma+1}_M$ at some point z_0 of $p^{-1}(0)$, then u belongs to $H^{s+\sigma+1}$ at all points of the bicharacteristic strip through z_0.*

2. Quasi-homogeneous pseudo-differential operators. Here, we list the facts on quasi-homogeneous pseudo-differential operators, which will be used in the proof of the theorem.

Let $p\in S^m_M(\Omega)$. Then $p(x,D_x)$ maps $\mathcal{E}'(\Omega)\cap H^s_M(\Omega)$ continuously into $H^{s-m}_{M,\mathrm{loc}}(\Omega)$. At non-characteristic points we obtain

Proposition 1. *Let $p\in S^m_M(\Omega)$ be a classical symbol. If $p(x,D_x)u\in H^s_M(z_0)$ and $p_m(z_0)\neq 0$, then $u\in H^{s+m}_M(z_0)$.*

The following proposition was proved by Lascar [3]. Here we shall reduce this to the setting of Proposition 3.5.1 of Hörmander [2] by proving "Sharp Gårding inequality" to quasi-homogeneous pseudo-differential operators.

Proposition 2. *Let $p\in S^m_M(\Omega)$ be a classical M-pseudo-differential operator with real principal symbol p_m and with simple characteristics (i.e. $H^M_p\neq 0$ on $p_m^{-1}(0)$). Let γ be a null bicharacteristic strip passing through z_0. If $u\in\mathcal{D}'(\Omega)$ satisfies $p(x,D_x)u\in H^s_M(\Omega)$ and $u\in H^{s+m-1}_M(z_0)$, then $u\in H^{s+m-1}_M(\gamma)$.*

Let $\nu=\inf\{\mu_j-1\}$. Notice that if $p,\,q\in S^m_M(\Omega)$ are classical M-pseudo-differential operators then

$$[p(x,D_x),q(x,D_x)]=-i\{p_m,q_m\}_M(x,D_x)+r(x,D_x)u,$$

where

$$\{p_m,q_m\}_M=\sum_{\{\mu_j=1\}}(\partial_{\xi_j}p_m\partial_{x_j}q_m-\partial_{x_j}p_m\partial_{\xi_j}q_m)=H^M_p q_m\in S^{m-1}_M(\Omega)$$

and $r(x,\xi)\in S^{m-1-\nu}_M(\Omega)$. Then Proposition 2 will be proved in the same way as in the proof of Proposition 3.5.1 of [2] with the aid of the following lemma.

Lemma 3. *Let $p\in S^m_M(\Omega)$ be a classical M-pseudo-differential operator and assume that*

$$\mathrm{Re}\,p_m(x,\xi)\geq 0.$$

Then for every $K\subset\subset\Omega$ there exists a constant C_K such that

$$\mathrm{Re}\,(p(x, D_x)u, u) \geq -C_K \|u\|^2_{M,(m-1)/2} \qquad \text{for } u \in C_0^\infty(K).$$

Proof. We shall prove the lemma by the method of wave packet transformation due to Cordoba-Fefferman [1]. Let us define the operator W: $\mathcal{E}'(\Omega) \cap L^2(\Omega) \to L^2_{\mathrm{loc}}(\Omega \times \boldsymbol{R}_n)$ by

$$Wu(y, \xi) = c_n [\xi]_M^{n/4} \int e^{i\langle y-x, \xi\rangle - [\xi]_M |y-x|^2/2} u(x)\,dx,$$

where $c_n = (2\pi)^{-3n/4}$ and let W^* be its adjoint:

$$W^*F(x) = c_n \iint e^{i\langle x-y, \xi\rangle - [\xi]_M |y-x|^2/2} [\xi]_M^{n/4} F(y, \xi)\,dy d\xi.$$

Then, we obtain that if $p \in S^m_M(\Omega)$ and $u \in C_0^\infty(\Omega)$ then

$$W^*pWu = p(x, D_x)u + q(x, D_x, x)u,$$

where q is a multiple symbol belonging to $S^{m-1}_{M,1,1/2}(\Omega)$, that is, for any $\alpha, \beta, \gamma \in \boldsymbol{N}^n$ it satisfies the estimate

$$|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma q(x, \xi, y)| (1+[\xi]_M)^{-m+1+\langle \alpha, M\rangle - (|\beta|+|\gamma|)/2} < \infty$$

locally uniformly in $x, y \in \Omega$. The lemma follows from the following two facts:

(1) $\int W^*pWu(x) \cdot \overline{u(x)}\,dx \geq 0$ when $p \geq 0$, which comes from

$$\int W^*pWu(x) \cdot \overline{u(x)}\,dx = \iint p(y, \xi) |Wu(y, \xi)|^2 dy d\xi, \text{ and}$$

(2) for $q \in S^{m-1}_{M,1,1/2}(\Omega)$ we have $|(q(x, D_x, x)u, u)| \leq C_K \|u\|^2_{M,(m-1)/2}$ for $u \in C_0^\infty(K)$.

3. Estimate to the non-linear term. In order to estimate a non-linear term micro-locally we prepare a lemma on the paraproduct, which was proved by Yamazaki [5].

Recall the definition of the paraproduct $\pi: S' \times S' \to S'$. If u and v are two tempered distributions, $\pi(u \cdot v)$ is defined by

$$\widehat{\pi(u \cdot v)}(\xi) = \int_{[\xi-\eta]_M \leq \varepsilon[\eta]_M} \hat{u}(\xi-\eta)\hat{v}(\eta)\,d\eta,$$

where ε is a small constant such that for $[\xi-\eta]_M \leq \varepsilon[\eta]_M$ there exists a constant $c > 0$ such that

$$\frac{1}{c}[\eta]_M \leq [\xi]_M \leq c[\eta]_M.$$

Lemma 4. *Let $F(x, u_1, \cdots, u_N)$ be a function which is holomorphic in $u_1, \cdots, u_n$ and C^∞ in x. Suppose that $f_1, \cdots, f_N \in H^s_M$ with $s > |M|/2$ and that they have values in the domain of definition of F. Then*

$$F(x, f_1(x), \cdots, f_N(x)) = \sum_{j=1}^N \pi\left(\frac{\partial F}{\partial u_j}(x, f_1(x), \cdots, f_N(x)), f_j\right) + G(x),$$

where $G \in H^{2s-|M|/2}_{M,\mathrm{loc}}$.

Applying this lemma to $f(u, \bar{u})$ we obtain

Corollary 5. *Let $f(u, \bar{u})$ be a holomorphic function of u, $\bar{u}$ and let $s > |M|/2$, $\sigma \leq s - |M|/2$. If $u \in H^s_{M,\mathrm{loc}}(\Omega) \cap H^{s+\sigma}_M(z_0) \cap H^{s+\sigma}_M(\check{z}_0)$ then $f(u, \bar{u}) \in H^s_{M,\mathrm{loc}}(\Omega) \cap H^{s+\sigma}_M(z_0) \cap H^{s+\sigma}_M(\check{z}_0)$, where $\check{z}_0$ denotes the anti-podal of z_0* (i.e. *if $z_0 = (x_0, \xi_0)$ then $\check{z}_0 = (x_0, -\xi_0)$*).

4. Proof of the theorem. Let γ be the bicharacteristic strip through z_0. First, notice that if $u \in H^s_{M,\mathrm{loc}}(\Omega)$ for $s > |M|/2$ Corollary 5 implies $f(u, \bar{u}) \in H^s_{M,\mathrm{loc}}(\Omega)$. From this and from $u \in H_M^{\min\{s+1, s+1+\sigma\}}(z_0)$ it follows that $u \in H_M^{\min\{s+1, s+1+\sigma\}}(\gamma)$ by Proposition 2. We have also $u \in H_M^{s+2}(\check{\gamma})$ by Proposition 1, because $\check{\gamma}$ consists of non-characteristic points. Again, Corollary 5 implies that

$$f(u, \bar{u}) \in H_M^{\min\{s+1, s+\sigma\}}(\gamma) \cap H_M^{\min\{s+1, s+\sigma\}}(\check{\gamma}).$$

Then by Propositions 1 and 2, it follows that

$$u \in H_M^{\min\{s+1, s+\sigma\}+1}(\gamma) \cap H_M^{\min\{s+1, s+\sigma\}+2}(\check{\gamma}).$$

If $s+\sigma < s+1$ we have done. If not, we can continue this process and conclude that $u \in H_M^{s+\sigma+1}(\gamma) \cap H_M^{s+\sigma+2}(\check{\gamma})$, which proves the theorem.

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