36. Fourier Transform of a Space of Holomorphic Discrete Series

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1. Let G be a connected non-compact real simple Lie group of matrices and K a maximal compact subgroup of G. Assume G/K is a hermitian symmetric space. Then, G/K can be realized as a Siegel domain D of type II. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}=\mathrm{Lie}\,G$ contained in $\mathfrak{k}=\mathrm{Lie}\,K$, Δ the root system of $(\mathfrak{g}_C, \mathfrak{h}_C)$. We introduce an order in Δ compatible with the complex structure of G/K. For each K-dominant K-integral linear form Λ on $\mathfrak{h}_C$ satisfying Harish-Chandra's non-vanishing condition [1, p. 612], the holomorphic discrete series Π_Λ of G is realized on a Hilbert space $\mathcal{H}(\Lambda)$ (see 5) of vector valued holomorphic functions on D. Let $S(D)$ be the Šilov boundary of D. Then, one knows that $S(D)$ is diffeomorphic to a certain nilpotent subgroup $N(D)$ of the affine automorphisms of D. By identifying $S(D)$ with $N(D)$, the aim of this note is a description of the space $\mathcal{H}(\Lambda)$ by using the Fourier transform on $N(D)$. If D reduces to a tube domain, $N(D)$ is abelian. Since such a description in this case is found in [6], we assume from now on that D does not reduce to a tube domain. Then, $N(D)$ is a simply connected 2-step nilpotent Lie group.

2. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ and $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$) the sum of all root subspaces corresponding to positive (resp. negative) non-compact roots in Δ. $\mathfrak{p}_\pm$ are abelian subalgebras of $\mathfrak{g}_C$ normalized by $\mathfrak{k}_C$. Let $P_\pm$ and K_C be analytic subgroups of G_C ($\mathrm{Lie}\,G_C=\mathfrak{g}_C$) corresponding to $\mathfrak{p}_\pm$ and $\mathfrak{k}_C$ respectively. Every $x\in P_+K_CP_-$ can be expressed in a unique way as $x=\exp\zeta_+\cdot k(x)\cdot\exp\zeta_-$ with $\zeta_\pm\in\mathfrak{p}_\pm$, $k(x)\in K_C$. We know that G is contained in $P_+K_CP_-$. Let $\{\gamma_1,\cdots,\gamma_l\}$ be a maximal system of positive non-compact strongly orthogonal roots such that for each j, γ_j is the largest positive non-compact root strongly orthogonal to $\gamma_{j+1},\cdots,\gamma_l$. For every $\alpha\in\Delta$, we choose $X_\alpha\in\mathfrak{g}_\alpha$ as in Lemma 3.1 in [2, p. 257]. Then,

$$\mathfrak{a}=\sum_{1\leqq i\leqq l}\mathbf{R}(X_{\gamma_i}+X_{-\gamma_i})$$

is a maximal abelian subspace of $\mathfrak{p}$ with l=real rank of G. Let

$$c=\exp\pi\sum_{1\leqq j\leqq l}(X_{\gamma_j}-X_{-\gamma_j})/4\in P_+K_CP_- \tag{1}$$

and $\nu=\mathrm{Ad}\,c$. As we are assuming that G/K does not reduce to a tube domain, there is only one possibility of positive $\mathfrak{a}$-root system $\Phi(\mathfrak{a})^+$ compatible with the original order in Δ through ν^* [3, p. 364]: put $2\lambda_j=\nu^*(\gamma_j)$, then

$$\Phi(\mathfrak{a})^+=\{\lambda_i+\lambda_j\,;\,1\leqq j\leqq i\leqq l\}\cup\{\lambda_i-\lambda_j\,;\,1\leqq j<i\leqq l\}\cup\{\lambda_i\,;\,1\leqq i\leqq l\}.$$

We denote by $\mathfrak{n}$ the sum of all positive $\mathfrak{a}$-root subspaces and put $\mathfrak{s}=\mathfrak{a}+\mathfrak{n}$. Let j be the complex structure on $\mathfrak{s}$ obtained by transforming the complex structure on $\mathfrak{p}$. We set

$$\mathfrak{s}(0)=\mathfrak{a}+\sum_{k<m}\mathfrak{n}_{\lambda_m-\lambda_k},\quad \mathfrak{s}(1/2)=\sum_{1\leqq k\leqq l}\mathfrak{n}_{\lambda_k},\quad \mathfrak{s}(1)=\sum_{k\leqq m}\mathfrak{n}_{\lambda_m+\lambda_k}.$$

Then, $\mathfrak{s}=\mathfrak{s}(0)+\mathfrak{s}(1/2)+\mathfrak{s}(1)$ and $\mathfrak{s}(0)$ is a subalgebra of $\mathfrak{g}$. Let $S(0)$ be the analytic subgroup of G corresponding to $\mathfrak{s}(0)$. Choose $s\in\mathfrak{s}(1)$ as in [7, p. 15] and let Ω be the $S(0)$-orbit of s in $\mathfrak{s}(1)$ under the adjoint representation. By [6, Theorem 4.15], Ω is a regular open convex cone in $\mathfrak{s}(1)$ and diffeomorphic to $S(0)$. For every $t\in\Omega$, we denote by $\eta_0(t)$ the unique element in $S(0)$ for which $(\mathrm{Ad}\,\eta_0(t))s=t$. On the other hand, it is known that $\mathfrak{s}(1/2)$ can be considered as a complex vector space V by $j|_{\mathfrak{s}(1/2)}$. Put $W=\mathfrak{s}(1)_C$. Then, $Q(x,y)=([jx,y]+i[x,y])/4$ is an Ω-positive hermitian map $V\times V\to W$. By using this pair of Ω and Q, we now define a Siegel domain D of type II: $D=\{(w,v)\in W\times V\,;\,\mathrm{Im}\,w-Q(v,v)\in\Omega\}$. Then, $S(D)=\{(x+iQ(\zeta,\zeta),\zeta)\,;\,x\in\mathfrak{s}(1),\ \zeta\in V\}$ and $N(D)=\{n(x,\zeta)\,;\,x\in\mathfrak{s}(1),\ \zeta\in V\}$ with multiplication

$$n(x,\zeta)n(x',\zeta')=n(x+x'+2\,\mathrm{Im}\,Q(\zeta,\zeta'),\ \zeta+\zeta').$$

3. Let Ξ be the set of all $\lambda\in\mathfrak{s}(1)^*$ such that the hermitian form $\lambda\circ Q$ is nondegenerate. Ξ contains the dual cone Ω^*. Now we have a family $(\pi_\lambda,\mathfrak{H}_\lambda)_{\lambda\in\Xi}$ of concrete irreducible unitary representations of $N(D)$ enough to decompose $L^2(N(D))$ (Kirillov model). For $\lambda\in\Xi$, let $\rho(\lambda)$ be the Pfaffian of the alternating bilinear form $\mathrm{Im}\,\lambda\circ Q$ on $\mathfrak{s}(1/2)$. The Fourier transform $\hat{f}$ of $f\in C_c^\infty(N(D))$ is by definition $\hat{f}(\lambda)=\int_{N(D)}f(n)\pi_\lambda(n^{-1})dn$, where dn is the Haar measure on $N(D)$. Then, the Plancherel formula for $N(D)$ is written as $\|f\|^2=C\int_\Xi\|\hat{f}(\lambda)\|_{\mathrm{HS}}^2\rho(\lambda)d\lambda$. The positive constant C depends only on the normalization of dn. One can define the Fourier transform of $f\in L^2(N(D))$ in the standard way.

4. Let $\psi\in C(\Omega)$ be everywhere positive such that $\psi(at)=a^\delta\psi(t)$ ($a>0$, $t\in\Omega$) for some $\delta\in\mathbf{R}$. Let $H^2(D,\psi)$ be the Hilbert space of $\mathbf{C}$-valued holomorphic functions on D satisfying

$$\|F\|^2=\int_D|F(x+iy,\zeta)|^2\,\psi(y-Q(\zeta,\zeta))dxdyd\zeta<\infty.$$

For $F\in H^2(D,\psi)$, put $f_t(x,\zeta)=F(x+i(t+Q(\zeta,\zeta)),\zeta)$ for every $t\in\Omega$. Then, f_t belongs to $L^2(N(D))$, so one can consider the Fourier transform $(f_t)^\wedge$. Now $\mathfrak{H}_\lambda$ can be identified with $L^2(\mathbf{R}^n)$, where $n=\dim_C V$. Let ϕ_0^λ be the zero-th Hermite function and V_λ the one dimensional subspace of $\mathfrak{H}_\lambda$ spanned by ϕ_0^λ. We denote by $\mathcal{H}^2(\Omega^*,\psi)$ the Hilbert space of functions Φ on Ξ taking value at $\lambda\in\Xi$ in the Hilbert space of Hilbert-Schmidt operators on $\mathfrak{H}_\lambda$ such that (i) $\Phi(\lambda)=0$ if $\lambda\notin\Omega^*$; (ii) $\mathrm{Range}\,\Phi(\lambda)\subset V_\lambda$ if $\lambda\in\Omega^*$; (iii) $\|\Phi\|^2=C\int_{\Omega^*}\|\Phi(\lambda)\|_{\mathrm{HS}}^2I_\psi(\lambda)\rho(\lambda)d\lambda<\infty$, where $I_\psi(\lambda)=\int_\Omega e^{-2\lambda(x)}\psi(x)dx$.

Theorem 1. *Let $F\in H^2(D,\psi)$ and f_t be as above. Then, $\Phi(\lambda)=e^{\lambda(t)}(f_t)^\wedge(\lambda)$ is independent of $t\in\Omega$ and belongs to $\mathcal{H}^2(\Omega^*,\psi)$. Conversely, let $\Phi\in\mathcal{H}^2(\Omega^*,\psi)$. Then,*

$$F(x+i(t+Q(\zeta,\zeta)),\zeta)=C\int_{\Omega^*}e^{-\lambda(t)}\,\mathrm{Tr}\,[\pi_\lambda(x,\zeta)\Phi(\lambda)]\rho(\lambda)d\lambda$$

is absolutely convergent and gives an element $F\in H^2(D,\psi)$ *such that* $\Phi(\lambda)=e^{\lambda(t)}(f_t)^{\wedge}(\lambda)$. *Moreover, the map* $F\mapsto\Phi$ *is unitary.*

5. Let Λ be as in 1 and τ_Λ the irreducible unitary representation of K on a finite dimensional Hilbert space E with highest weight Λ. Since P_+K_C is a semidirect product, τ_Λ can be naturally extended to a representation of P_+K_C. Let $c\in G_C$ be the element defined by (1) and put $\Phi_\Lambda(g)=\tau_\Lambda(k(c)^{-1})\tau_\Lambda(k(cg))$. We note $cg\in P_+K_CP_-$ for $g\in G$. Put

$$\theta_0(t)=|\det_{\mathfrak{s}(1/2)}\mathrm{Ad}\,\eta_0(t)|^{-1}|\det_{\mathfrak{s}(1)}\mathrm{Ad}\,\eta_0(t)|^{-2}\qquad(t\in\Omega)$$

and $\Theta_\Lambda(\alpha(h))=\Phi_\Lambda(h)$ ($h\in S=\exp\mathfrak{s}$), where α is the map $G\to D$ which induces a G-equivariant biholomorphism of G/K onto D. Now, $\mathcal{H}(\Lambda)$ consists of E-valued holomorphic functions on D with

$$\|F\|^2=\int_D\|\Theta_\Lambda(iy,\zeta)^{-1}F(x+iy,\zeta)\|^2\theta_0(y-Q(\zeta,\zeta))dxdyd\zeta<\infty.$$

Let v_Λ be a highest weight vector for τ_Λ normalized so that $\|v_\Lambda\|=1$. We take an orthonormal basis $v_1=v_\Lambda, v_2,\cdots,v_d$ ($d=\deg\tau_\Lambda$) in E consisting of weight vectors arranged in order so that any vector in the root subspaces corresponding to positive compact roots in Δ is represented, under τ_Λ, by an upper triangular matrix. We denote by Λ_j the weight for v_j. Let E_k be the one dimensional subspace of E spanned by v_k and $\mathcal{H}_j(\Lambda)=\{F\in\mathcal{H}(\Lambda)\,;\,F(w,\zeta)\in E^j\}$, where $E^j=E_1\oplus\cdots\oplus E_j$. Then, $\mathcal{H}_j(\Lambda)$ is a closed subspace of $\mathcal{H}(\Lambda)$ invariant under $\Pi_\Lambda|_S$. Let $\mathcal{H}^1(\Lambda)=\mathcal{H}_1(\Lambda)$ and $\mathcal{H}^j(\Lambda)=$ the orthogonal complement of $\mathcal{H}_{j-1}(\Lambda)$ in $\mathcal{H}_j(\Lambda)$ ($j\geqq2$). Put $Y_i=X_{\gamma_i}+X_{-\gamma_i}$ and define a positive character χ_j of $A=\exp\mathfrak{a}$ by $\chi_j(\exp\sum a_iY_i)=\prod\exp a_i\Lambda_j(\nu(Y_i))$. Extending χ_j canonically to a character of S, we put $\psi_j(t)=\chi_j(\eta_0(t))^{-2}\theta_0(t)$ for $t\in\Omega$. Then, $\psi_j(at)=a^{\delta_j}\psi_j(t)$ ($a>0$, $t\in\Omega$) for some $\delta_j\in\mathbf{R}$. Consider the Hilbert space $H^2(D,\psi_j)$ of the type in 4 and define an operator T_j by $T_jF(w,\zeta)=(F(w,\zeta),v_j)$ ($F\in\mathcal{H}_j(\Lambda)$). T_j is a bounded operator $\mathcal{H}_j(\Lambda)\to H^2(D,\psi_j)$ with dense range. Therefore by considering the polar decomposition of T_j, $\mathcal{H}^j(\Lambda)$ is unitarily isomorphic to $H^2(D,\psi_j)$. Thus we have an irreducible decomposition $\mathcal{H}(\Lambda)=\mathcal{H}^1(\Lambda)\oplus\cdots\oplus\mathcal{H}^d(\Lambda)$ for $\Pi_\Lambda|_S$ [6, p. 381].

6. Put $I_\Lambda(\lambda)=\int_\Omega e^{-2\lambda(t)}\Phi_\Lambda(\eta_0(t)^{-1})^2\theta_0(t)dt$ ($\lambda\in\Omega^*$). The integral is absolutely convergent. Now the matrix of $I_\Lambda(\lambda)$ relative to the basis (v_k) is upper triangular with (k,k)-entry $I_{\psi_k}(\lambda)>0$. Therefore we can give a meaning to $I_\Lambda(\lambda)^{1/2}$. Let $\boldsymbol{B}_2(\mathfrak{H}_\lambda)$ be the Hilbert space of Hilbert-Schmidt operators on $\mathfrak{H}_\lambda$. We put $A(\mathfrak{H}_\lambda)=\{T\in\boldsymbol{B}_2(\mathfrak{H}_\lambda)\,;\,\mathrm{Range}\,T\subset V_\lambda\}$. It is evident that $A(\mathfrak{H}_\lambda)$ is a closed subspace of $\boldsymbol{B}_2(\mathfrak{H}_\lambda)$. Consider the Hilbert space tensor product $A(\mathfrak{H}_\lambda)\otimes E$ of two Hilbert spaces $A(\mathfrak{H}_\lambda)$ and E. This is regarded as the Hilbert space of anti-linear Hilbert-Schmidt operators mapping E to $A(\mathfrak{H}_\lambda)$ via $(T\otimes v)(u)=(v,u)T$. For $\lambda\in\Omega^*$, we define an operator $M_\Lambda(\lambda)$ on $A(\mathfrak{H}_\lambda)\otimes E$ by $M_\Lambda(\lambda)(T\otimes v)=T\otimes I_\Lambda(\lambda)^{1/2}v$. Let $\boldsymbol{H}(\Lambda)$ be the Hilbert space of functions Ψ on Ξ whose value at $\lambda\in\Xi$ is in $A(\mathfrak{H}_\lambda)\otimes E$ such that

(i) $\Psi(\lambda)=0$ if $\lambda \notin \Omega^*$; (ii) $\|\Psi\|^2=C\int_{\Omega^*}\|M_A(\lambda)\Psi(\lambda)\|^2\rho(\lambda)d\lambda<\infty$.

Put $\boldsymbol{H}_j(\Lambda)=\{\Psi \in \boldsymbol{H}(\Lambda);\Psi(\lambda)\in \boldsymbol{A}(\mathfrak{H}_\lambda)\otimes E^j\}$ and $\boldsymbol{T}_j\Psi(\lambda)=\Psi(\lambda)v_j \in \boldsymbol{A}(\mathfrak{H}_\lambda)$ for $\Psi \in \boldsymbol{H}_j(\Lambda)$. $\boldsymbol{T}_j$ is a bounded operator $\boldsymbol{H}_j(\Lambda)\to\mathcal{H}^2(\Omega^*, \psi_j)$ with dense range. Let $\boldsymbol{H}^1(\Lambda)=\boldsymbol{H}_1(\Lambda)$ and $\boldsymbol{H}^j(\Lambda)$=the orthogonal complement of $\boldsymbol{H}_{j-1}(\Lambda)$ in $\boldsymbol{H}_j(\Lambda)$ $(j\geqq 2)$. Then, $\boldsymbol{H}^j(\Lambda)$ is unitarily isomorphic to $\mathcal{H}^2(\Omega^*, \psi_j)$ via the polar decomposition of $\boldsymbol{T}_j$. Therefore, we have an orthogonal decomposition $\boldsymbol{H}(\Lambda)=\boldsymbol{H}^1(\Lambda)\oplus\cdots\oplus\boldsymbol{H}^d(\Lambda)$. In view of Theorem 1, we get

Theorem 2. *$\mathcal{H}(\Lambda)$ is unitarily isomorphic to $\boldsymbol{H}(\Lambda)$ under the procedure described above.*

References

[1] Harish-Chandra: Amer. J. Math., **78**, 564–628 (1956).
[2] S. Helgason: Differential Geometry, Lie Groups and Symmetric Spaces. Academic Press, New York (1978).
[3] C. C. Moore: Amer. J. Math., **86**, 358–378 (1964).
[4] T. Nomura: A description of a space of holomorphic discrete series by means of the Fourier transform on the Šilov boundary (preprint).
[5] R. D. Ogden and S. Vagi: Adv. Math., **33**, 31–92 (1979).
[6] H. Rossi and M. Vergne: J. Funct. Anal., **13**, 324–389 (1973).
[7] M. Vergne and H. Rossi: Acta Math., **136**, 1–59 (1976).