34. *On the Cauchy Problem for Effectively Hyperbolic Systems*

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1. Introduction. In this note, we study the Cauchy problem for the first order hyperbolic systems which are effectively hyperbolic, that is the determinant of the principal part of these systems is effectively hyperbolic. Using the same method in [4], [6], we shall show that the Cauchy problem for such systems is C^∞ well posed for any choice of lower order terms, and give a theorem of propagation of wave front sets which is analogous to the result in [5], [7] for effectively hyperbolic operators.

In what follows, we use the following notation,

$$x=(x_0, x'), \quad x'=(x_1, \cdots, x_d), \quad \xi=(\xi_0, \xi'), \quad \xi'=(\xi_1, \cdots, \xi_d),$$
$$D_j=-i(\partial/\partial x_j), \qquad D'=(D_1, \cdots, D_d).$$

We study the Cauchy problem for $L(x, D)$ with the symbol

$$L(x, \xi)=\sum_{j=0}^{m} A_j(x, \xi')\xi_0^{m-j}, \qquad A_0(x, \xi')=I_N, \tag{1.1}$$

where $A_j(x, \xi')$ are $(N\times N)$ classical pseudo-differential symbols of degree j defined in a conic neighborhood W of $(0, \bar{\xi}')$ in the cotangent bundle $T^*\boldsymbol{R}^d$, depending smoothly on x_0 in an open interval I containing the origin and I_N is the identity matrix of degree N.

Denote by $h(x, \xi)$ the determinant of the principal part $L_0(x, \xi)$ of $L(x, \xi)$. We shall say that $L_0(x, \xi)$ is effectively hyperbolic system at $\rho=(0, \bar{\xi}) \in \boldsymbol{R}\times T^*\boldsymbol{R}^d$ if $h(x, \xi)$ satisfies the following conditions,

(1.2) $h(x, \xi)$ is hyperbolic with respect to dx_0, that is the equation $h(x, \xi_0, \xi')=0$ has only real roots in ξ_0 for any (x, ξ') near ρ,

(1.2) if $dh(0, \xi_0, \bar{\xi}')=0$, then the fundamental matrix $F_h(0, \xi_0, \bar{\xi}')$ has non zero real eigenvalues,

(for the definition of the fundamental matrix, see [1], [2]). Then we have

Theorem 1.1. *Assume that $m=1$ and $L_0(x, \xi)$ is effectively hyperbolic at $\rho=(0, \bar{\xi}')$. Then, in a sufficiently small conic neighborhood of $(0, \bar{\xi}') \in T^*\boldsymbol{R}^d$, there is a parametrix of $L(x, D)$ with finite propagation speed of wave front sets.*

From this Theorem, it follows that

Theorem 1.2. *Assume that $m=1$ and $L_0(x, \xi)$ is effectively hyperbolic system at every $(0, \xi')$ $(|\xi'|=1)$. Then the Cauchy problem for $L(x, D)$ is locally solvable in the C^∞ class in a neighborhood of the origin in $\boldsymbol{R}^{d+1}$ with the data on $x_0=0$.*

Remark 1.1. Parametrices in conic open sets with finite propagation speed of WF are defined in [4] for scalar operators. A generalization of the definition to systems of the form (1.1) is almost obvious.

Let $h(\sigma)=0$, $\sigma=(0,\bar{\xi})\in T^*\boldsymbol{R}^{d+1}$. Denote by h_σ the lowest homogeneous part in the Taylor expansion of $h(x,\xi)$ at σ. It is well known that h_σ is a hyperbolic polynomial on $T(T^*\boldsymbol{R}^{d+1})$. We denote by $\Gamma(h,\sigma)$ the component of $\theta=(0,1,\cdots,0)$ in $\{(x,\xi)\,;\,h_\sigma(x,\xi)\neq 0\}$ and by H_ϕ the Hamilton vector field of ϕ. For a vector distribution $u=(u_1,\cdots,u_N)$, we set

$$WF(u)=\bigcup_{i=1}^{N} WF(u_i).$$

Then we have

Theorem 1.3. *Assume that $m=1$ and $L_0(x,\xi)$ is effectively hyperbolic system at $\rho=(0,\bar{\xi}')$. Let $\phi(x,\xi)$ be a real smooth function near σ such that $\phi(\sigma)=0$, $-H_\phi(\sigma)\in\Gamma(h,\sigma)$ and Ω be a sufficiently small conic neighborhood of σ in $T^*\boldsymbol{R}^{d+1}$. Then it follows from*

$$\sigma\notin WF(Lu),\qquad \Omega\cap WF(u)\cap\{\phi(x,\xi)<0\}=\emptyset$$

that

$$\sigma\notin WF(u),$$

where u is a vector distribution.

2. Reduction to the case $m=2$, $N=2$. Since the multiplicities of ξ_0-roots of the equation $h(x,\xi)=0$ are at most 2, it is routine to find an $(N\times N)$ pseudo-differential symbol $T(x,\xi')$ such that near ρ (see for example [8]),

$$L(x,D)T(x,D')=T(x,D')\tilde{L}(x,D)$$

modulo $C^\infty(I_0,S^{-\infty})$ and $\tilde{L}(x,\xi)=L_1(x,\xi)\oplus\cdots\oplus L_s(x,\xi)$ where I_0 is an open interval containing the origin and $L_j(x,\xi)$ has the form (1.1) with $m=1$, $N=2$ or $m=1$, $N=1$. If every $L_j(x,D)$ has a parametrix in Γ with finite propagation speed of WF then so does $\tilde{L}(x,D)$ in Γ. Hence $L(x,D)$ has a parametrix with finite propagation speed of WF in $\tilde{\Gamma}$ for any $\tilde{\Gamma}\Subset\Gamma$. Now suppose that $L(x,\xi)$ has the form (1.1) with $m=1$, $N=2$ and denote by ${}^{co}L_0(x,\xi)$ the cofactor matrix of $L_0(x,\xi)$. If $L(x,D)\,{}^{co}L_0(x,D)$ has a parametrix with finite propagation speed of WF in Γ then so does $L(x,D)$ in Γ. Then the existence of such parametrix is reduced to that of $L(x,D)\,{}^{co}L_0(x,D)$. Here we note that

$$L(x,D)\,{}^{co}L_0(x,D)=p(x,D)I_2+B(x,D)\tag{2.1}$$

where $p(x,\xi)=\det L_0(x,\xi)$ and $B(x,\xi)=B_0(x,\xi')\xi_0+B_1(x,\xi')$ with (2×2) pseudo-differential symbols $B_i(x,\xi')$ of degree i. We remark that $p(x,\xi)$ is effectively hyperbolic at ρ.

Since the existence of parametrix is invariant under (scalar) elliptic Fourier integral operators on $\boldsymbol{R}^d$, then we may assume that

$$p(x,\xi)=\xi_0^2-Q(x,\xi')$$

with $Q(x,\xi')$ satisfying one of the following two conditions ([7]).

$$Q(x,\xi')\geqq c(x_0-\phi(x',\xi'))^2|\xi'|^2,\quad \{\phi,\{\phi,Q\}\}(\rho)=0,\quad \rho=(0,e_1),\tag{2.2}$$

$$\begin{gathered}Q(x,\xi')=M(x,\xi')^2+\tilde{Q}(x,\xi'),\qquad \tilde{Q}(x,\xi')\geqq c(x_0-\phi(x',\xi'))^2|\xi'|^2,\\ \{\phi,\{\phi,\tilde{Q}\}\}(\rho)=0,\quad |\{\phi,M\}(\rho)|<1,\quad \rho=(0,e_2)\end{gathered}\tag{2.3}$$

where e_p denotes the unit vector in $\boldsymbol{R}^d$ with the p-th component 1.

3. Microlocal energy estimates for localized systems. In this section, we study the system (2.1) with $p(x, \xi)=\xi_0^2-Q(x, \xi')$ where $Q(x, \xi')$ satisfies either (2.2) or (2.3). Following [6], [7], introducing a small positive parameter μ, we define the localization $L(x, \xi, \mu)$ of $L(x, \xi)$,

$$L(x, D, \mu)=p(x, D, \mu)I_2+B_0(x, D', \mu)D_0+B_1(x, D', \mu),$$
$$p(x, D, \mu)=D_0^2-Q(x, D', \mu)$$

where $Q(x, \xi', \mu)$, $B_i(x, \xi', \mu)$ are localizations of $Q(x, \xi')$, $B_i(x, \xi')$ respectively which are defined for any $(x', \xi') \in \boldsymbol{R}^{2d}$, $x_0 \in I_1$, $0<\mu \leqq \bar{\mu}$. We notice that $L(x, \xi, \mu)$ coincides with $\mu^2 L(\mu x^{(1)}, \mu^{1/2}x^{(2)}, \mu^{-1}\xi^{(1)}, \mu^{-1/2}\xi^{(2)})$ in a conic neighborhood of ρ (which depends on μ) where $x^{(1)}=(x_0, \cdots, x_p)$, $x^{(2)}=(x_{p+1}, \cdots, x_d)$, $\xi^{(1)}=(\xi_0, \cdots, \xi_p)$, $\xi^{(2)}=(\xi_{p+1}, \cdots, \xi_d)$ with $p=1$ or 2 according to the case (2.2) or (2.3).

To formulate the energy estimates for $L(x, D, \mu)$, we introduce some symbols.

$$J_\pm(x, \xi', \mu)=\pm\{2\chi_0(\pm(x_0-\mu^{-1}\phi(x', \xi', \mu))\langle\mu\xi'\rangle^{1/2})-1\}$$
$$\times(x_0-\mu^{-1}\phi(x', \xi', \mu))+\langle\mu\xi'\rangle^{-1/2},$$
$$\alpha_n^\pm(x, \xi', \mu)=\chi(\pm n^{1/2}(x_0-\mu^{-1}\phi(x', \xi', \mu))\langle\mu\xi'\rangle^{1/2}), \qquad \langle\xi'\rangle^2=1+\sum_{j=1}^d \xi_j^2,$$
$$I_\pm(n, r)(x, \xi', \mu)=\langle\mu\xi'\rangle^{\max(0, \mp n)}J_\pm(x, \xi', \mu)^{\mp n-r},$$

where $\chi_0(s)$, $\chi(s)$ are smooth functions on $\boldsymbol{R}$ such that

$\chi_0(s)=0$ for $s\leqq -1/2$, $\chi_0(s)=1$ for $s\geqq -1/4$, $0\leqq\chi_0(s)\leqq 1$ for $s\in\boldsymbol{R}$,
$\chi(s)=0$ for $s\leqq -1$, $\chi(s)=1$ for $s\geqq 1$, $\chi(s)+\chi(-s)=1$ for $s\in\boldsymbol{R}$,

and $\phi(x', \xi', \mu)$ is the localization of $\phi(x', \xi')$ in (2.2)–(2.3). Using the operators with above symbols we define the following semi-norms.

$$|||u|||_{n,r,k}^2=\sum_{i=1}^2 \sum{}^\pm \|\langle\mu D'\rangle^k I_\pm(n, r)\alpha_n^\pm u_i\|^2, \quad \|u\|_k^2=\sum_{i=1}^2 \|u_i\|_k^2, \quad u=(u_1, u_2)$$

where $\|\cdot\|$ stands for the L^2-norm on $\boldsymbol{R}^d$ and $\|\cdot\|_k$ denotes the usual Sobolev norm in $H^k(\boldsymbol{R}^d)$. Then we have for $L_\theta(x, D, \mu)=L(x, D_0-i\theta, D', \mu)$ that

Theorem 3.1. *For any* $u\in C_0^\infty(I_2\times\boldsymbol{R}^d)$, $n\geqq n_0$, $0<\mu\leqq\mu_0(n)$, $\theta\geqq\theta_0(n, \mu, s, k)$, $s, k\in\boldsymbol{R}$ *with* $k\geqq 1-s$, *we have*

$$\int |||L_\theta u|||_{n,0,s}^2 dx_0+c(n, \mu, s, k)\int \|L_\theta u\|_{-k}^2 dx_0$$
$$\geqq c_1 n\int (|||(D_0-i\theta)u|||_{n,1,s}^2+|||u|||_{n,0,s+1}^2)dx_0$$
$$+c_2\theta\int (|||(D_0-i\theta)u|||_{n,1/2,s}^2+|||u|||_{n,-1/2,s+1}^2)dx_0$$
$$+c_3\theta\int (\|(D_0-i\theta)u\|_{-k}^2+\theta\|u\|_{-k}^2)dx_0.$$

From this Theorem, the existence of parametrix (in a sufficiently small conic neighborhood of $(0, \bar{\xi}')\in T^*\boldsymbol{R}^d$) follows easily. To show that this parametrix has finite propagation speed of WF, we prepare one lemma. From now on, we fix n, μ so that the estimates in Theorem 3.1 hold. To simplify notation, we set

$$E_{n,s}^2(u; x_0)=|||u|||_{n,0,s}^2+|||(D_0-i\theta)u|||_{n,1,s-1}^2+\theta|||u|||_{n,-1/2,s}^2+\theta|||(D_0-i\theta)u|||_{n,1/2,s-1}^2.$$

Let $\phi(x, \xi')$ be a real smooth function, positively homogeneous of degree 0. We define $\Phi(x, \xi')$ by ([3])

(3.1) $\Phi(x, \xi')=\exp(-1/\phi(x, \xi'))$ if $\phi(x, \xi')>0$ and 0 if $\phi(x, \xi')\leqq 0$.
Then we have

Lemma 3.1. *Assume that $\phi(x, \xi')$ satisfies*

$$4(1-\nu)Q(x, \xi', \mu)\geqq\{Q, \phi\}^2(x, \xi', \mu), \qquad (\partial/\partial x_0)\phi(x, \xi')=-1,$$

with some positive ν. Then for any $\theta\geqq\theta_0(s, k)$, $s, k\in \boldsymbol{R}$ with $k\geqq 1-s$, we have

$$c(s)\int |||\Phi L_\theta u|||^2_{n,0,s}\,dx_0+c(s, k)\int \|\Phi L_\theta u\|^2_{-k}\,dx_0+c(s, \nu)\int E^2_{n,s+3/4}(u\,;x_0)dx_0$$
$$+\theta\int (\|(D_0-i\theta)u\|^2_{-k}+\theta\|u\|^2_{-k})dx_0\geqq\int E^2_{n,s+1}(\Phi u\,;x_0)dx_0,$$

where Φ is defined by (3.1) *with above ϕ and $\{\ ,\ \}$ denotes the Poisson bracket.*

Using this Lemma, we can show that the parametrix assured by Theorem 3.1 has finite propagation speed of *WF* following [4]. This Lemma and the same arguments as in [5] show Theorem 1.3 also.

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