58. Local Isometric Embedding of Two Dimensional Riemannian Manifolds into R^3 with Nonpositive Gaussian Curvature

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As it is well known, the problem of C^∞ local isometric embedding of a two dimensional Riemannian manifold into $\boldsymbol{R}^3$ is a problem whether C^∞ functions $x(u, v)$, $y(u, v)$, $z(u, v)$ which satisfy

(1) $$dx^2+dy^2+dz^2=Edu^2+2Fdudv+Gdv^2$$

exist in a neighborhood of a point, say $(u, v)=0$, when the first fundamental form $Edu^2+2Fdudv+Gdv^2$ is given. The results already known are as follows. Let K be the Gaussian curvature of the two dimensional manifold, then the classical result is that the problem is affirmatively answered if $K\neq 0$ at $(u, v)=0$, and a recent interesting result due to Lin [3] is that it is also affirmative if $K=0$, $grad\, K\neq 0$ at $(u, v)=0$. Now a natural question arises. Namely, is it affirmative when $K=grad\, K=0$ at $(u, v)=0$ and one of the following conditions holds:

(i) $Hess\, K(0, 0)>0$,

(ii) $Hess\, K(0, 0)<0$,

(iii) $Hess\, K(0, 0)$ has two eigenvalues with opposite signs?

Hereafter, for simplicity, we refer to the case with conditions $K=grad\, K=0$ at $(u, v)=0$ and (i) (resp. (ii) and resp. (iii)) by (i) (resp. (ii) and resp. (iii)).

Then what we have obtained is the following.

Theorem. *The problem of C^∞ local isometric embedding is also affirmative in the case* (ii).

The idea of the proof is as follows. Since a two dimensional Riemannian manifold whose Gaussian curvature is zero is locally isometric to Euclidean space with its standard metric, it is enough to solve the following equation (2) for z under the condition $\nabla z(0, 0)=0$, which assures the Gaussian curvature of the metric

$$Edu^2+2Fdudv+Gdv-dz^2$$

vanishes in a neighborhood of $(u, v)=0$. Namely,

(2) $$(z_{11}-\Gamma^i_{11}z_i)(z_{22}-\Gamma^i_{22}z_i)-(z_{12}-\Gamma^i_{12}-z_i)^2$$
$$=K(EG-F^2-Ez_2^2-Gz_1^2+2Fz_1z_2)$$

where Γ^i_{jk} are Christoffel symbols, z_i is the first derivative of z with respect to the i-th variable and z_{jk} is the second derivative with respect to the j-th and k-th variables by calling u the first variable and v the second variable.

Now we construct an approximate solution $\bar{z}$ which satisfies (2) modulo a certain term with flatness $O(u^4)$ and linearize (2) at $\bar{z}$. Then the linearized equation of (2) becomes an effectively hyperbolic equation with a time variable u. By using a symbol calculus, we can show that this effectively hyperbolic equation is stable under certain flat perturbation and satisfies an energy equality of evolutional type.

Next we introduce two kinds of Banach scales with norms

$$|z|_m = \sup_{0<|z|\le T} \Sigma_{j=0}^{3} |u|^{-5+j} \|(\partial/\partial u)^j z\|_{m+2-j+\overline{m}},$$

$$[z]_m = \sup_{0<|u|\le T} \Sigma_{j=0}^{1} |u|^{-3+j} \|(\partial/\partial u)^j z\|_{m+\overline{m}-j}$$

where $\overline{m}$ is some fixed positive costant, and $\|\cdot\|_s$ denotes the Sobolev norm of order s with respect to $v \in \boldsymbol{R}^1$. Then, according to the above two facts for our effectively hyperbolic equation, we can compensate $\bar{z}$ to find a true solution of (2) by applying Nash's implicit function theorem in the framework of the two Banach scales with norms $|\cdot|_m$ and $[\cdot]_m$.

Remark. One can ask the local existence of a nonparametric hypersurface with prescribed Gaussian curvature K under the same condition on K, and obtain an affirmative answer for the case (ii). Moreover, it (resp. Theorem) is also affirmative if $K=-LM^l$ with $l \in N$ (resp. $l=2$) and C^∞ functions K, L defined in a neighborhood of the origin such that $L>0$, $M=0$, $grad\, M=0$, $Hess\, M>0$ at the origin. These results can be proved by a similar method which is used to prove our theorem.

The detailed proof will be published elsewhere.

References

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