84. The Fourier-Borel Transformations of Analytic Functionals on the Complex Sphere

By Ryoko WADA
Sophia University

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1. Introduction. Let d be a positive integer and $d\geqslant 2$. $S=S^d$ denotes the unit sphere in $\boldsymbol{R}^{d+1}$. $L(z)$ and $L^*(z)$ denote the Lie norm and the dual Lie norm on $\boldsymbol{C}^{d+1}$ respectively:

$$L(z)=L(x+iy)=[\|x\|^2+\|y\|^2+2\{\|x\|^2\|y\|^2-(x\cdot y)^2\}^{1/2}]^{1/2},$$
$$L^*(z)=\sup\{|\xi\cdot z|;\ L(\xi)\leqslant 1\},$$

where $\xi\cdot z=\sum_{j=1}^{d+1}\xi_j\cdot z_j$, $x, y\in\boldsymbol{R}^{d+1}$, and $\|x\|^2=x\cdot x$.

$\mathcal{O}(\boldsymbol{C}^{d+1})$ denotes the space of entire functions on $\boldsymbol{C}^{d+1}$. We put

$$\mathrm{Exp}\,(\boldsymbol{C}^{d+1}:(r:N))=\lim_{r'>r}\mathrm{proj}\ X_{r':N}\qquad\text{for } 0\leqslant r<\infty$$

and

$$\mathrm{Exp}\,(\boldsymbol{C}^{d+1}:[r:N])=\lim_{r'<r}\mathrm{ind}\ X_{r':N}\qquad\text{for } 0<r\leqslant\infty,$$

where N is a norm on $\boldsymbol{C}^{d+1}$ and

$$X_{r':N}=\{f\in\mathcal{O}(\boldsymbol{C}^{d+1});\ \sup_{z\in\boldsymbol{C}^{d+1}}|f(z)|e^{-r'N(z)}<\infty\}.$$

We denote the complex sphere by $\tilde{S}=\{z\in\boldsymbol{C}^{d+1};\ z_1^2+z_2^2+\cdots+z_{d+1}^2=1\}$, and we put $\tilde{S}(r)=\{z\in\tilde{S};\ L(z)<r\}$ for $r>1$ and $\tilde{S}[r]=\{z\in\tilde{S};\ L(z)\leqslant r\}$ for $r\geqslant 1$. $\mathcal{O}(\tilde{S}(r))$ denotes the space of holomorphic functions on $\tilde{S}(r)$ and we put $\mathcal{O}(\tilde{S}[r])=\lim\mathrm{ind}_{r'>r}\,\mathcal{O}(\tilde{S}(r'))$. $\mathrm{Exp}\,(\tilde{S})$ denotes the restriction to $\tilde{S}$ of the space $\mathrm{Exp}\,(\boldsymbol{C}^{d+1})$ of entire functions of exponential type. $\mathcal{O}'(\tilde{S}(r))$, $\mathcal{O}'(\tilde{S}[r])$ and $\mathrm{Exp}'\,(\tilde{S})$ denote the dual spaces of $\mathcal{O}(\tilde{S}(r))$, $\mathcal{O}(\tilde{S}[r])$, and $\mathrm{Exp}\,(\tilde{S})$ respectively.

The *Fourier-Borel transformation* P_λ for a functional $f'\in\mathrm{Exp}'\,(\tilde{S})$ is defined by

$$P_\lambda f'(z)=\langle f'_\xi,\ \exp i\lambda(\xi\cdot z)\rangle\qquad\text{for } z\in\boldsymbol{C}^{d+1},$$

where $\lambda\in\boldsymbol{C}$, $\lambda\neq 0$ is a fixed constant.

Morimoto [1] determined the images of $\mathrm{Exp}'\,(\tilde{S})$ and $\mathcal{O}'(\tilde{S})$ by P_λ. The purpose of this paper is to determine the images of $\mathcal{O}'(\tilde{S}(r))$ and $\mathcal{O}'(\tilde{S}[r])$ by P_λ.

2. Statement of results. Our main theorem in this paper is following

Theorem 2.1. *P_λ establishes the following linear topological isomorphisms*:

(2.1) $\qquad P_\lambda:\mathcal{O}'(\tilde{S}(r))\xrightarrow{\sim}\mathrm{Exp}_\lambda\,(\boldsymbol{C}^{d+1}:[|\lambda|r:L^*])\qquad(r>1),$

(2.2) $\qquad P_\lambda:\mathcal{O}'(\tilde{S}[r])\xrightarrow{\sim}\mathrm{Exp}_\lambda\,(\boldsymbol{C}^{d+1}:(|\lambda|r:L^*))\qquad(r\geqslant 1),$

where $\mathrm{Exp}_\lambda\,(\boldsymbol{C}^{d+1}:[|\lambda|r:L^*])=\mathcal{O}_\lambda(\boldsymbol{C}^{d+1})\cap\mathrm{Exp}\,(\boldsymbol{C}^{d+1}:[|\lambda|r:L^*])$, $\mathrm{Exp}_\lambda\,(\boldsymbol{C}^{d+1}:(|\lambda|r:L^*))=\mathcal{O}_\lambda(\boldsymbol{C}^{d+1})\cap\mathrm{Exp}\,(\boldsymbol{C}^{d+1}:(|\lambda|r:L^*))$, *and* $\mathcal{O}_\lambda(\boldsymbol{C}^{d+1})=\{f\in\mathcal{O}(\boldsymbol{C}^{d+1});\ (\Delta_z+\lambda^2)f(z)=0\}$.

Let $M=\{z\in \boldsymbol{C}^{d+1};\ z_1^2+z_2^2+\cdots+z_{d+1}^2=0,\ z\neq 0\}$. We define

$$Ff'(z)=\langle f'_\xi, e^{\xi\cdot z}\rangle \qquad \text{for } z\in M.$$

Ff' is the restriction of $P_{-i}f'$ to M. We put

$$\mathrm{Holo}\,(M)=\mathcal{O}(\boldsymbol{C}^{d+1})|_M,$$
$$\mathrm{Exp}\,(M, r/\sqrt{2})=\bigcap_{r'>(r/\sqrt{2})}\{\psi\in \mathrm{Holo}\,(M);\ \sup_{z\in M}|\psi(z)|e^{-r'\|z\|}<\infty\},$$
$$\mathrm{Exp}\,[M, r/\sqrt{2}]=\bigcup_{r'<(r/\sqrt{2})}\{\psi\in \mathrm{Holo}\,(M);\ \sup_{z\in M}|\psi(z)|e^{-r'\|z\|}<\infty\},$$

and

$$\mathrm{Exp}\,(M)=\mathrm{Exp}\,[M,\infty], \quad \text{and} \quad \mathcal{J}(M)=\{f\in\mathcal{O}(\boldsymbol{C}^{d+1});\ f=0 \text{ on } M\},$$

where $\|z\|^2=\sum_{j=1}^{d+1}|z_j|^2$.

The topologies of Holo (M), Exp (M), Exp $(M, r/\sqrt{2})$ and Exp $[M, r/\sqrt{2}]$ are defined to be the quotient topologies $\mathcal{O}(\boldsymbol{C}^{d+1})/\mathcal{J}(M)$, $\mathrm{Exp}\,(\boldsymbol{C}^{d+1})/(\mathcal{J}(M)\cap \mathrm{Exp}\,(\boldsymbol{C}^{d+1}))$, $\mathrm{Exp}\,(\boldsymbol{C}^{d+1}:(r:L^*))/(\mathcal{J}(M)\cap\mathrm{Exp}\,(\boldsymbol{C}^{d+1}:(r:L^*)))$, and $\mathrm{Exp}\,(\boldsymbol{C}^{d+1}:[r:L^*])/(\mathcal{J}(M)\cap\mathrm{Exp}\,(\boldsymbol{C}^{d+1}:[r:L^*]))$ respectively. Then we have

Theorem 2.2. *The transformation $F: f'\to\langle f'_\xi, e^{\xi\cdot z}\rangle$ establishes the following linear topological isomorphisms:*

(2.3) $F:\mathrm{Exp}'(\tilde{S})\xrightarrow{\sim}\mathrm{Holo}\,(M)$,

(2.4) $F:\mathcal{O}'(\tilde{S})\xrightarrow{\sim}\mathrm{Exp}\,(M)$,

(2.5) $F:\mathcal{O}'(\tilde{S}[r])\xrightarrow{\sim}\mathrm{Exp}\,(M, r/\sqrt{2})\qquad (r\geqslant 1)$,

(2.6) $F:\mathcal{O}'(\tilde{S}(r))\xrightarrow{\sim}\mathrm{Exp}\,[M, r/\sqrt{2}]\qquad (r>1)$.

Corollary 2.3. i) *For any $f\in\mathcal{O}(\boldsymbol{C}^{d+1})$ there exists a unique $g\in\mathcal{O}_\lambda(\boldsymbol{C}^{d+1})$ such that $f=g$ on M.*

ii) *For any $f\in\mathcal{O}(\boldsymbol{C}^{d+1})$ such that $\sup_{z\in M}|f(z)|e^{-A\|z\|}<\infty$ for some $A>0$, there exists a unique $g\in\mathrm{Exp}_\lambda(\boldsymbol{C}^{d+1})$ such that $f=g$ on M.*

iii) *For any $f\in\mathcal{O}(\boldsymbol{C}^{d+1})$ such that $\sup_{z\in M}|f(z)|\exp(-|\lambda|r'\|z\|/\sqrt{2})<\infty$ for $\forall r'>r$, there exists a unique $g\in\mathrm{Exp}_\lambda(\boldsymbol{C}^{d+1}:(|\lambda|r:L^*))$ such that $f=g$ on M.*

iv) *For any $f\in\mathcal{O}(\boldsymbol{C}^{d+1})$ such that $\sup_{z\in M}|f(z)|\exp(-|\lambda|r'\|z\|/\sqrt{2})<\infty$ for some $r'<r$, there exists a unique $g\in\mathrm{Exp}_\lambda(\boldsymbol{C}^{d+1}:[|\lambda|r:L^*])$ such that $f=g$ on M.*

3. Outline of the proof of the results. We put $N=\{z=x+iy\in M;\ \|x\|=\|y\|=1\}$. ds and dN denote the unique $O(d+1)$ invariant measures on S and N respectively. $\|\ \|_2$ and $\|\ \|_N$ denote the L^2-norms on S and N with $\|1\|_2=\|1\|_N=1$ respectively. $H_{n,d}$ is the space of spherical harmonics of degree n in $(d+1)$ dimensions and $P_n(M)$ is the restriction to M of the space of homogeneous polynomials of degree n on $\boldsymbol{C}^{d+1}$. In order to prove our theorems we need following lemmas.

Lemma 3.1. i) *F is a one-to-one linear mapping of $H_{n,d}$ onto $P_n(M)$ and we have for $f\in H_{n,d}$*

(3.1) $\|f\|_2=C_n^{1/2}\|Ff\|_N,\qquad$ *where* $C_n=\dfrac{n!\,\Gamma(n+(d+1)/2)}{\Gamma((d+1)/2)}\dim H_{n,d}$.

ii) *If ψ_n belongs to $P_n(M)$ and ψ_l belongs to $P_l(M)$ and $n\neq l$, we have*

$$\int_N \psi_n(z)\overline{\psi_l(z)}\,dN=0.$$

Outline of the proof. i) If we denote $P_{n,d}$ the Legendre polynomial of degree n and of dimension $d+1$, $\{P_{n,d}(\ \cdot\alpha);\ \alpha\in S\}$ spans $H_{n,d}$. For $f=P_{n,d}(\ \cdot\alpha)$ we have $Ff(z)=(n!\dim H_{n,d})^{-1}(z\cdot\alpha)^n$, which shows $F(H_{n,d})\subset P_n(M)$. Since $\dim H_{n,d}=\dim P_n(M)$, F is surjective. It is valid that

$$\int_S P_{n,d}(s\cdot\alpha)\overline{P_{n,d}(s\cdot\beta)}ds=C'_n\int_N (z\cdot\alpha)^n\overline{(z\cdot\beta)^n}dN,$$

where

$$C'_n=\Gamma\left(n+\frac{d+1}{2}\right)\bigg/\left(\Gamma\left(\frac{d+1}{2}\right)n!\dim H_{n,d}\right),$$

so we have (3.1) for $\forall f\in H_{n,d}$ and (3.1) implies that F is injective.

ii) Since $\int_N (z\cdot\alpha)^n\overline{(z\cdot\beta)^l}dN=0$ we can prove ii). Q.E.D.

Lemma 3.2. *F is a one-to-one linear mapping of* $\mathrm{Exp}'(\tilde{S})$ *onto* $\mathrm{Holo}(M)$, $\mathcal{O}'(\tilde{S})$ *onto* $\mathrm{Exp}(M)$, $\mathcal{O}'(\tilde{S}[r])$ *onto* $\mathrm{Exp}(M, r/\sqrt{2})$, *and* $\mathcal{O}'(\tilde{S}(r))$ *onto* $\mathrm{Exp}[M, r/\sqrt{2}]$.

Outline of the proof. From [1] Theorem 4.1 (Martineau's theorem) and Theorem 7.1 we have $F(\mathrm{Exp}'(\tilde{S}))\subset\mathrm{Holo}(M)$, $F(\mathcal{O}'(\tilde{S}))\subset\mathrm{Exp}(M)$, $F(\mathcal{O}'(\tilde{S}[r]))\subset\mathrm{Exp}(M, r/\sqrt{2})$, and $F(\mathcal{O}'(\tilde{S}(r)))\subset\mathrm{Exp}[M, r/\sqrt{2}]$. For all $\psi\in\mathrm{Holo}(M)$ there exist $\psi_n\in P_n(M)$ $(n=0, 1, 2, \cdots)$ such that $\psi=\sum_{n=0}^{\infty}\psi_n$. By Lemma 3.1 there exist $f_n\in H_{n,d}$ $(n=0, 1, \cdots)$ such that $\psi_n=Ff_n$ and $\|f_n\|_2\leqslant\sqrt{C_n}K_n$, where $K_n=\sup_{z\in N}|\psi_n(z)|$. If ψ belongs to $\mathrm{Holo}(M)$ (resp. $\psi\in\mathrm{Exp}(M)$, $\psi\in\mathrm{Exp}(M, r/\sqrt{2})$, $\psi\in\mathrm{Exp}[M, r/\sqrt{2}]$) we have

$$\limsup_{n\to\infty} K_n^{1/n}=0$$

(resp. $K_n\leqslant C(\sqrt{2}Ae/n)^n$ for some $A>0$, $K_n\leqslant C_{r'}(r'e/n)^n$ for $\forall r'>r$, $K_n\leqslant C'(r'e/n)^n$ for some $r'<r$, where $C, C_{r'}, C'$ are constants). From these facts and [1] Theorem 6.1, if we put $f'=\sum_{n=0}^{\infty}f_n$ we get $f'\in\mathrm{Exp}'(\tilde{S})$ (resp. $\mathcal{O}'(\tilde{S})$, $\mathcal{O}'(\tilde{S}[r])$, $\mathcal{O}'(\tilde{S}(r))$) and $Ff'=\psi$. The injectivity of F is proved by Lemma 3.1. Q.E.D.

Proof of Theorem 2.1. From [1] Theorems 4.1 and 7.1 we have $P_\lambda(\mathcal{O}'(\tilde{S}(r)))\subset\mathrm{Exp}_\lambda(C^{d+1}:[|\lambda|r:L^*])$ and $P_\lambda(\mathcal{O}'(\tilde{S}[r]))\subset\mathrm{Exp}_\lambda(C^{d+1}:(|\lambda|r:L^*))$ and P_λ is injective. Let $\tilde{\psi}$ be in $\mathrm{Exp}_\lambda(C^{d+1}:[|\lambda|r:L^*])$, $\tilde{\psi}|_M=\psi$ and $\psi_{1/i\lambda}(z)=\psi(z/\lambda i)$. Then $\psi_{1/i\lambda}\in\mathrm{Exp}[M, r/\sqrt{2}]$ and there exists $f'\in\mathcal{O}'(\tilde{S}(r))$ such that $Ff'=\psi_{1/i\lambda}$ from Lemma 3.2. On the other hand, from [1] Theorem 7.1 there exists $h'\in\mathrm{Exp}'(\tilde{S})$ such that $\tilde{\psi}=P_\lambda h'$. For any $z\in M$ we have $Fh'(z)=\tilde{\psi}(z/i\lambda)=\psi(z/i\lambda)$, so we get $Ff'=Fh'$ and $f'=h'$ by Lemma 3.2. From [1] Theorem 4.1 and the closed graph theorem P_λ and P_λ^{-1} are continuous. Then we obtain (2.1). Similarly we can prove (2.2). Q.E.D.

Theorem 2.2 follows from Theorem 2.1 and Lemma 3.2. From [1] Theorem 7.1 and Theorems 2.1 and 2.2 we obtain Corollary 2.3.

Full details will appear elsewhere. The author would like to thank Professor M. Morimoto for his helpful suggestions.

Reference

[1] M. Morimoto: Analytic functionals on the sphere and their Fourier-Borel transformations, Complex Analysis, Banach Center Publications, **11**, PWN-Polish Scientific Publishers, Warsaw, 223–250 (1983).