55. On Some Integral Invariants on Complex Manifolds. I

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This note is a continuation of our preceding works (cf. Bando [1], Mabuchi [8]) and we here explain how Futaki invariants (cf. Futaki [5], Futaki and Morita [6]) are generalized and reinterpreted from our viewpoints. Most of the proofs down below are very sketchy and a complete account including the present results will be given in a separate paper [2].

(I) Fix an arbitrary compact complex r-dimensional connected manifold X. Let $G:=\mathrm{Aut}(X)$ be the group of all holomorphic automorphisms of X and $G^0:=\mathrm{Aut}^0(X)$ be its identity component. We denote by $\mathcal{V}_X$ the set of all volume forms Ω on X such that $\int_X \Omega=1$. Now, to each pair $(\Omega', \Omega'')\in\mathcal{V}_X\times\mathcal{V}_X$, we associate the real number $N_X(\Omega', \Omega'')\in\boldsymbol{R}$ by

$$N_X(\Omega', \Omega''):=\int_a^b dt\int_X \{(\sqrt{-1}/2\pi)\bar{\partial}\partial\log(\Omega_t)\}^r(\partial\Omega_t/\partial t)/\Omega_t,$$

where $\{\Omega_t\,|\,a\leqq t\leqq b\}$ is an arbitrary piecewise smooth path in $\mathcal{V}_X$ such that $\Omega_a=\Omega'$ and $\Omega_b=\Omega''$. Then by a result of Donaldson [4; Proposition 6] applied to the anti-canonical bundle K_X^{-1} of X, the number $N_X(\Omega', \Omega'')$ above is independent of the choice of the path $\{\Omega_t\,|\,a\leqq t\leqq b\}$ and therefore well-defined. Furthermore, N_X is G-invariant, i.e.,

$$N_X(g^*\Omega', g^*\Omega'')=N_X(\Omega', \Omega'') \qquad \text{for all } g\in G \text{ and all } \Omega', \Omega''\in\mathcal{V}_X,$$

and satisfies the 1-cocycle condition, i.e.,

(i) $N_X(\Omega', \Omega'')+N_X(\Omega'', \Omega')=0$ and

(ii) $N_X(\Omega, \Omega')+N_X(\Omega', \Omega'')+N_X(\Omega'', \Omega)=0$,

for all $\Omega, \Omega', \Omega''\in\mathcal{V}_X$. We now fix an arbitrary element Ω_0 of $\mathcal{V}_X$, and define a functional $\nu_X:\mathcal{V}_X\to\boldsymbol{R}$ by

$$\nu_X(\Omega):=N_X(\Omega_0, \Omega), \qquad \Omega\in\mathcal{V}_X.$$

We moreover set

$$n_X(g):=\exp(\nu_X(g^*\Omega_0)), \qquad g\in G.$$

Then the same argument as in [8; § 5] easily allows us to obtain:

Proposition A. (i) *$n_X: G\to\boldsymbol{R}_+$ is a Lie group homomorphism which does not depend on the choice of Ω_0, where $\boldsymbol{R}_+$ denotes the group of positive real numbers. In particular, n_X is trivial on $[G, G]$.*

(ii) *Let $\lambda:=c_1(X)^r[X]$. Then $\Omega\in\mathcal{V}_X$ is a critical point of ν_X if and only if* $\{(\sqrt{-1}/2\pi)\bar{\partial}\partial\log(\Omega)\}^r=\lambda\Omega$, i.e., $(\sqrt{-1}/2\pi)\bar{\partial}\partial\log(\Omega)$ *is a (possibly indefinite) Einstein form.*

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(iii) *Let H be a subgroup of G such that both $H \supset G^0$ and $[H : G^0] < +\infty$ are satisfied. Then if ν_X has a critical point on $\mathcal{V}_X$, the restriction $n_{X|H}$ of n_X to H is trivial. In particular, if K_X^{-1} is ample and ν_X has a critical point on $\mathcal{V}_X$, then n_X is trivial.*

Note that the twistor space of a compact quaternionic Kähler manifold of negative scalar curvature is a compact complex manifold with a natural indefinite Einstein form (cf. Salamon [9]).

(II) We next assume that X is a compact connected r-dimensional Kähler manifold with Kähler form ω_0, where ω_0 is so normalized that $\int_X \omega_0^r = 1$. We then put

$$\mathcal{K} := \{\text{Kähler forms on } X \text{ cohomologous to } \omega_0\},$$
$$\omega_0(\psi) := \omega_0 + \sqrt{-1}\partial\bar{\partial}\psi, \qquad \psi \in C^\infty(X)_{\boldsymbol{R}},$$
$$\mathcal{H} := \{\psi \in C^\infty(X)_{\boldsymbol{R}} \mid \omega_0(\psi) \in \mathcal{K}\}.$$

For each holomorphic vector field $v \in \Gamma(X, \mathcal{O}(TX))$ on X, we denote by L_v (resp. ∇_v) the Lie derivative (resp. covariant derivative in terms of the Kähler metric ω_0) with respect to v, where we use Kähler forms and the corresponding Kähler metrics interchangeably. Furthermore, for each $\omega \in \mathcal{K}$, let $c_i(\omega)$ be the i-th Chern form of the Kähler metric ω. Then the "Futaki invariants" of X are regarded as the linear map

$$F\langle c_1^{r+1}\rangle : \Gamma(X, \mathcal{O}(TX)) \longrightarrow \boldsymbol{R}$$

defined by

$$F\langle c_1^{r+1}\rangle(v) := 2\,\mathrm{Re}\int_X \mathrm{Tr}\,(L_v - \nabla_v)c_1(\omega_0)^r, \qquad v \in \Gamma(X, \mathcal{O}(TX)),$$

where $\mathrm{Tr}\,(L_v - \nabla_v) \in C^\infty(X)_{\boldsymbol{C}}$ denotes the trace of the C^∞ section $L_v - \nabla_v$ of the vector bundle $\mathrm{End}\,(TX)$ over X (cf. Futaki and Morita [6], Berline and Vergne [3]). Now, in view of a theorem of Lichnerowicz (see for instance [7; p. 94]), we can without difficulty show that:

Proposition B. (i) *$-F\langle c_1^{r+1}\rangle$ is nothing but the Lie algebra homomorphism associated with the Lie group homomorphism $n_X : G \to \boldsymbol{R}_+$.*

(ii) *$\tilde{\omega} \in \mathcal{K}$ is a critical point for the functional $\mathcal{K} \ni \omega \mapsto \nu_X(\omega^r) \in \boldsymbol{R}$ if and only if $\tilde{\omega}$ is an Einstein-Kähler form.*

Fix an arbitrary $p \in \boldsymbol{Z}$ with $0 \leq p \leq r$ and let

$$\lambda_p := \int_X c_p(X) \wedge \omega_0^{r-p}.$$

Now, to each pair $(\omega', \omega'') \in \mathcal{K} \times \mathcal{K}$, we associate a real number $M_p(\omega', \omega'')$ by

$$M_p(\omega', \omega'') := \int_a^b \left\{\int_X (\partial\psi_t/\partial t)(c_p(w_t) \wedge \omega_t^{r-p} - \lambda_p\omega_t^r)\right\} dt,$$

where $\{\psi_t \mid a \leq t \leq b\}$ is an arbitrary piecewise smooth path in $\mathcal{H}$ such that $\omega_t := \omega_0(\psi_t)$ satisfies the boundary conditions $\omega_a = \omega'$ and $\omega_b = \omega''$. Then

Theorem C. *$M_p(\omega', \omega'')$ above is independent of the choice of the path $\{\psi_t \mid a \leq t \leq b\}$ and therefore well-defined. Furthermore, M_p is G-invariant and also satisfies the 1-cocycle condition.*

Proof. Let $\{\psi_{s,t}\}$ be a smooth two-parameter family of functions $\psi_{s,t}$ in $\mathcal{H}$, and $\theta_{s,t}$ be the curvature form of the Kähler metric $\omega_{s,t} := \omega_0(\psi_{s,t})$.

Then $c_p(\omega_{s,t})$ is written as $P_{s,t}:=P(\theta_{s,t})$ for some invariant polynomial P of degree p. Now, in view of [8; § 2], it suffices to show

$$\int_X (\partial\psi_{s,t}/\partial s)(\partial/\partial t)\{P_{s,t}\wedge\omega_{s,t}^{r-p}\}=\int_X (\partial\psi_{s,t}/\partial t)(\partial/\partial s)\{P_{s,t}\wedge\omega_{s,t}^{r-p}\}.$$

But then, this easily follows from integral by parts by virtue of an argument in [1].

Definition. We define a subgroup $G_{\mathcal{K}}$ of G $(=\mathrm{Aut}(X))$ by

$$G_{\mathcal{K}}:=\{g\in \mathrm{Aut}(X)\,|\,g^*\mathcal{K}=\mathcal{K}\}\qquad (\supset G^0).$$

Furthermore, put as follows:

$$\mu_p(\omega):=M_p(\omega_0,\omega)\qquad \omega\in\mathcal{K},$$
$$m_p(g):=\exp(\mu_p(g^*\omega_0)),\qquad g\in G_{\mathcal{K}}.$$

Now, just by the same argument as in deriving Proposition A, we obtain:

Theorem D. (i) *$m_p:G_{\mathcal{K}}\to\boldsymbol{R}_+$ is a Lie group homomorphism which does not depend on the choice of ω_0. In particular, m_p is trivial on $[G_{\mathcal{K}},G_{\mathcal{K}}]$.*

(ii) *$\tilde{\omega}\in\mathcal{K}$ is a critical point for the functional $\mu_p:\mathcal{K}\to\boldsymbol{R}$ if and only if $c_p(\tilde{\omega})\wedge\tilde{\omega}^{r-p}=\lambda_p\tilde{\omega}^r$.*

(iii) *Let H be a subgroup of $G_{\mathcal{K}}$ such that both $H\supset G^0$ and $[H:G^0]<+\infty$ are satisfied. Then if μ_p has a critical point on $\mathcal{K}$, the restriction $m_{p|H}$ of m_p to H is trivial. In particular, if K_X^{-1} is ample and μ_p has a critical point on $\mathcal{K}$, then m_p is trivial.*

Let $H\phi_0$ be the harmonic part of $\phi_0:=c_p(\omega_0)$ in terms of the Kähler metric ω_0. Then there exists a real C^∞ $(p-1,p-1)$-form F_0 an X such that

$$\phi_0-H\phi_0=\sqrt{-1}\partial\bar{\partial}F_0.$$

We now put (cf. Bando [1])

$$\beta_p(v):=\mathrm{Re}\int_X (L_vF_0)\wedge\omega_0^{r-p+1},\qquad v\in\Gamma(X,\mathcal{O}(TX)).$$

In view of [1] and [8; § 5], one immediately sees that:

Theorem E. *$\{-2/(r-p+1)\}\beta_p:\Gamma(X,\mathcal{O}(TX))\to\boldsymbol{R}$ is nothing but the Lie algebra homomorphism associated with the Lie group homomorphism $m_p:G_{\mathcal{K}}\to\boldsymbol{R}_+$.*

Note that the $\mathcal{K}$-energy map defined in [8] coincides with μ_1 above up to constant multiple and that β_1 is the "Futaki invariants" of X.

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