85. On the Existence of Polyharmonic Functions in Lebesgue Classes

By Yoshihiro MIZUTA
Faculty of Integrated Arts and Sciences, Hiroshima University

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For a domain D in the n-dimensional euclidean space R^n, we denote by $H_m(D)$ the class of all functions polyharmonic of order m in D. We say that a subspace $\mathcal{H}$ of $H_m(D)$ is non-trivial if it has an element which is not extended to a function polyharmonic of order m in R^n. Our aim in this note is to find a condition for $\mathcal{H}$ to be non-trivial. This problem is closely related to the removable singularities for polyharmonic functions (see Adams-Polking [1], Harvey-Polking [3], [4], Maz'ja-Havin [5], Mizuta [7]).

The Bessel capacity of index (α, p) of a set E is defined by

$$B_{\alpha,p}(E)=\inf \int |f(y)|^p \, dy,$$

where the infimum is taken over all nonnegative measurable functions f such that $g_\alpha * f(x) \geqq 1$ for all $x \in E$, g_α being the Bessel kernel of order α (cf. Meyers [6]).

Theorem 1. *Let m be a positive integer, $1<p<\infty$, $1/p+1/q=1$ and D be a domain in R^n.*

(i) *If $B_{2m,p}(R^n-D)=0$, then $H_m(D)\cap L^q(D)=\{0\}$.*

(ii) *If $2mp\leqq n$ and $B_{2m,p}(R^n-D)>0$, then $H_m(D)\cap L^q(D)$ is non-trivial.*

(iii) *If $2m-n/p$ is a positive number which is not an integer and R^n-D contains at least two points, then $H_m(D)\cap L^q(D)$ is non-trivial.*

(iv) *If $2m-n/p$ is a positive integer and R^n-D contains three distinct points x_1, x_2, x_3 such that $2x_2=x_1+x_3$, then $H_m(D)\cap L^q(D)$ is non-trivial.*

Proof. Statement (i) is an easy consequence of [1; Theorem B] and the fact that $H_m(R^n)\cap L^q(R^n)=\{0\}$.

Assume that the conditions in (ii) are satisfied. Then we can find mutually disjoint compact subsets $K_1, K_2 \subset R^n-D$ such that $B_{2m,p}(K_i)>0$ for $i=1,2$. By [6; Theorem 16], there exist nonnegative measures μ_1, μ_2 such that the support of μ_i is included in K_i, $\mu_i(K_i)=1$ and $g_{2m}*\mu_i \in L^q(R^n)$ for each i. Consider the function

$$u(x)=\int |x-y|^{2m-n} \, d\mu_1(y)-\int |x-z|^{2m-n} \, d\mu_2(z).$$

Since $g_m*\mu_i \in L^q(R^n)$, $u \in L^q_{\mathrm{loc}}(R^n)$. Further, noting that $u(x)=O(|x|^{2m-n-1})$ as $|x|\to\infty$, we can prove that $u \in L^q(R^n)$. Clearly, $u \in H_m(D)$ and u is not extended to a function polyharmonic of order m in R^n. Thus assertion (ii) is proved.

Assume $2mp>n$, and let l be the nonnegative integer such that $l\leqq 2m-n/p<l+1$. For a multi-index α, we set

$$v_\alpha(x)=(D^\alpha R_{2m})(x+x_0)-\sum_{|\beta|\leqq l-|\alpha|}(x_0^\beta/\beta!)(D^{\alpha+\beta}R_{2m})(x),$$

where R_{2m} denotes the Riesz kernel of order $2m$, $D^\alpha=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$, $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ and $\alpha!=\alpha_1!\cdots\alpha_n!$ for a point $x=(x_1,\cdots,x_n)$ and a multi-index $\alpha=(\alpha_1,\cdots,\alpha_n)$. If $l<2m-n/p$ and $R^n-D\supset\{x_0,0\}$, $x_0\neq 0$, then $v_\alpha\in H_m(D)\cap L^q(D)$ for each α with $|\alpha|\leqq l$. If $l=2m-n/p$ and $R^n-D\supset\{x_0,0,-x_0\}$, $x_0\neq 0$, then we see from the mean value theorem that

$$\begin{aligned}w_\alpha(x)=&(D^\alpha R_{2m})(x+x_0)-\sum_{|\beta|\leqq l-1-|\alpha|}(x_0^\beta/\beta!)(D^{\alpha+\beta}R_{2m})(x)\\&-(-1)^{l-|\alpha|}[(D^\alpha R_{2m})(x-x_0)-\sum_{|\beta|\leqq l-1-|\alpha|}((-x_0)^\beta/\beta!)(D^{\alpha+\beta}R_{2m})(x)]\end{aligned}$$

belongs to $H_m(D)\cap L^q(D)$ for each α with $|\alpha|\leqq l-1$. By a change of coordinate system, statements (iii) and (iv) are shown to be true.

Remark. Let $2mp>n$, and l be a nonnegative integer such that $l\leqq 2m-n/p<l+1$.

(i) If $l<2m-n/p$ and $R^n-D=\{x_0,0\}$, $x_0\neq 0$, then
$$H_m(D)\cap L^q(D)=\{\sum_{|\alpha|\leqq l}a_\alpha v_\alpha\,;\,a_\alpha\in R^1\}.$$

(ii) If $l<2m-n/p$ and R^n-D consists of one point, then
$$H_m(D)\cap L^q(D)=\{0\}.$$

(iii) If $l=2m-n/p$ and $R^n-D=\{x_0,0,-x_0\}$, $x_0\neq 0$, then
$$H_m(D)\cap L^q(D)=\{\sum_{|\alpha|\leqq l-1}a_\alpha w_\alpha\,;\,a_\alpha\in R^1\}.$$

(iv) Suppose $l=2m-n/p$ and R^n-D consists of three points x_1,x_2,x_3. Then $H_m(D)\cap L^q(D)$ is non-trivial if and only if $2x_2=x_1+x_3$, $2x_3=x_1+x_2$ or $2x_1=x_2+x_3$ holds.

Here v_α and w_α are the functions defined in the proof of Theorem 1.

For a proof of the Remark, it suffices to use the following elementary fact (cf. [10; Théorème XXXIV 2°]).

Lemma. *Let u be a tempered distribution in R^n such that $\Delta^m u=0$ on $R^n-\{x_1,x_2,\cdots,x_k\}$. Then u is of the form*
$$u(x)=\sum_{i,\alpha}c_{i,\alpha}D^\alpha R_{2m}(x-x_i)+P(x),$$
where $c_{i,\alpha}\in R^1$ and P is a polynomial which is polyharmonic of order m in R^n.

We denote by $BL_k(L^q(D))$ the space of all functions $u\in L^q_{\mathrm{loc}}(D)$ such that $D^\alpha u\in L^q(D)$ for any multi-index α with $|\alpha|=k$, where the derivatives are taken in the sense of distributions (cf. [2]). It is noted that if $u\in H_m(R^n)\cap BL_k(L^q(R^n))$, then u is a polynomial of degree at most $k-1$, since $D^\alpha u=0$ for any multi-index α with $|\alpha|=k$ by the above lemma.

Theorem 2. *Let m,p,q and D be as in Theorem 1. Let k be a positive integer which is not larger than $2m$.*

(i) *If $B_{2m-k,p}(R^n-D)=0$, then each function in $H_m(D)\cap BL_k(L^q(D))$ is a polynomial of degree at most $k-1$, where $B_{0,p}$ denotes the n-dimensional Lebesgue measure.*

(ii) *If* $(2m-k)p \leqq n$ *and* $B_{2m-k,p}(R^n - D) > 0$, *then* $H_m(D) \cap BL_k(L^q(D))$ *is non-trivial.*

(iii) *If* $2m-k-n/p$ *is a non-integral positive number and* $R^n - D$ *contains at least two points, then the same conclusion as in* (ii) *holds.*

(iv) *If* $2m-k-n/p$ *is a positive integer and* $R^n - D$ *contains three distinct points* x_1, x_2, x_3 *such that* $2x_2 = x_1 + x_3$, *then the same conclusion as in* (ii) *holds.*

Proof. We shall prove (ii) only, because the remaining part can be proved in the same manner as in the proof of Theorem 1.

Assume that the conditions in (ii) are fulfilled. As in the proof of Theorem 1, we can find mutually disjoint compact sets $K_1, K_2 \subset R^n - D$ and nonnegative measures μ_1, μ_2 such that the support of μ_i is included in K_i, $\mu_i(K_i) = 1$ and $g_{2m-k} * \mu_i \in L^q(R^n)$ for each i, where g_0 denotes the dirac measure at the origin. Set

$$f_\alpha(y) = \int D^\alpha R_{2m}(y-z)d\mu_1(z) - \int D^\alpha R_{2m}(y-z)d\mu_2(z)$$

and

$$u(x) = \sum_{|\alpha|=k, |\beta|=2m-k} a_{\alpha,\beta} \int K_{m,\beta}(x,y) f_\alpha(y)dy,$$

where

$$K_{m,\beta}(x,y) = D^\beta R_{2m}(x-y) \qquad \text{if } |y|<1,$$
$$K_{m,\beta}(x,y) = D^\beta R_{2m}(x-y) - \sum_{|\gamma| \leqq l'} (x^\gamma/\gamma!)(D^{\beta+\gamma} R_{2m})(-y) \qquad \text{if } |y| \geqq 1,$$

l' being the integer such that $l' \leqq k - n/q < l'+1$, and $a_{\alpha,\beta}$ are constants so chosen that $\sum_{|\alpha|=k, |\beta|=2m-k} a_{\alpha,\beta} D^{\alpha+\beta}$ is the Laplace operator iterated m times. Since $g_{2m-k} * \mu_i \in L^q(R^n)$, $f_\alpha \in L^q_{\mathrm{loc}}(R^n)$ for $|\alpha| = k$. Further, noting that $f_\alpha(y) = O(|y|^{2m-k-n-1})$ as $|y| \to \infty$, we see that $f_\alpha \in L^q(R^n)$. Hence, as in the proof of [8; Lemma 3], it follows that $u \in BL_k(L^q(R^n))$. By use of Fubini's theorem, we can show that $\Delta^m u = c(\mu_1 - \mu_2)$ in the sense of distributions, where c is a non-zero constant. Thus $u \in H_m(D) \cap BL_k(L^q(D))$. Finally, noting that u can not be extended to a function polyharmonic of order m in R^n, we end the proof of statement (ii).

Remark 1. Let $(2m-k)p > n$ and l be the nonnegative integer such that $l \leqq 2m-k-n/p < l+1$.

(i) If $l < 2m-k-n/p$ and $D = R^n - \{x_0, 0\}$, $x_0 \neq 0$, then

$$H_m(D) \cap BL_k(L^q(D)) = \{\sum_{|\alpha| \leqq l} a_\alpha v_\alpha + P;\ a_\alpha \in R^1,\ P \in H_m(R^n) \cap P_{k-1}\},$$

where P_{k-1} denotes the family of polynomials of degree at most $k-1$.

(ii) If $l = 2m-k-n/p$ and $D = R^n - \{x_0, 0, -x_0\}$, $x_0 \neq 0$, then

$$H_m(D) \cap BL_k(L^q(D)) = \{\sum_{|\alpha| \leqq l-1} a_\alpha w_\alpha + P;\ a_\alpha \in R^1,\ P \in H_m(R^n) \cap P_{k-1}\}.$$

Here v_α and w_α are the functions defined in the proof of Theorem 1.

Remark 2. If $R^n - D$ has positive n-dimensional Lebesgue measure, then, as in Nguyen [9], we can find a function in $H_m(D)$ whose derivatives of order $2m-1$ are Lipschitzian.

References

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