95. *Uniqueness of the ω-limit Point of Solutions of a Semilinear Heat Equation on the Circle*

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§ 1. **Introduction.** In this paper we discuss the asymptotic behavior of a semilinear heat equation on the circle. Namely we shall show that the ω-limit set of any solution contains at most one element. This implies that any bounded global solution converges to an equilibrium solution as $t\to\infty$.

To be more precise, consider the initial value problem

$$(1.1)\qquad \begin{cases} u_t=u_{xx}+f(u), & x\in \boldsymbol{R},\ t>0,\\ u(x,0)=u_0(x), & x\in \boldsymbol{R},\\ u_0(x+1)=u_0(x), & x\in \boldsymbol{R},\end{cases}$$

where f is a C^1-function on $\boldsymbol{R}$ and u_0 is a continuous function of period 1. It is well known that (1, 1) has a unique classical solution $u(x,t)$ such that $u(\cdot,t)\in L^\infty(\boldsymbol{R})$ for every $t\in[0,s(u))$, where $[0,s(u))$ denotes the maximal interval of existence for the solution u. Also one can easily show that $u(x+1,t)=u(x,t)$ for $x\in\boldsymbol{R}$, $t>0$. Therefore (1.1) can be regarded as an equation on a circle.

For a solution u of (1.1), its ω-limit set is defined by

$$(1.2)\qquad \omega(u)=\bigcap_{0<t<s(u)} \text{closure}\ \{u(\cdot,\tau):\tau>t\},$$

where the "closure" is with respect to the topology of $C^2(\boldsymbol{R})$.

A standard *a priori* estimate and dynamical systems argument show that $\omega(u)\neq\varnothing$ if and only if $s(u)=\infty$ and there exists a sequence $t_n\to\infty$ such that $u(\cdot,t_n)$ remains bounded as $n\to\infty$, in which case $\omega(u)$ is a connected locally compact subset of $C^2(\boldsymbol{R})$. Using the Lyapunov functional, one can show that each $\phi\in\omega(u)$ is an equilibrium of (1.1), that is, ϕ satisfies

$$(1.3)\qquad \begin{cases} \phi''+f(\phi)=0, & x\in\boldsymbol{R},\\ \phi(x+1)=\phi(x), & x\in\boldsymbol{R}.\end{cases}$$

For the details, see, for instance, [1] and [4].

For each $\lambda\in\boldsymbol{R}$, we let $\phi(x;\lambda)$ be the solution of the following initial value problem for ordinary differential equation:

$$(1.4)\qquad \begin{cases} \phi''+f(\phi)=0, & x\in\boldsymbol{R},\\ \phi(0;\lambda)=\lambda, \\ \phi'(0;\lambda)=0,\end{cases}$$

where $'$ stands for d/dx. Define

$$\Lambda(f)=\{\lambda\in\boldsymbol{R}:\phi(x;\lambda)\text{ is nonconstant and 1-periodic in }x\}.$$

Our main result is the following:

Theorem. *If $\Lambda(f)$ contains no interior point, then $\omega(u)$ contains at* *most one element.*

A direct consequence is

Corollary. *If $\Lambda(f)$ contains no interior point and u is a solution of* (1.1), *then either* $\lim_{t\to s(u)}\|u(\cdot,t)\|_{L^\infty(\boldsymbol{R})}=\infty$, *or there exists a solution $\phi(x)$ of* (1.3) *such that* $\lim_{t\to\infty}u(\cdot,t)=\phi$ *in $C^2(\boldsymbol{R})$, with $s(u)=\infty$.*

In his pioneering work [3], Matano obtained an analogous (and stronger) result in the case of initial-boundary value problems on a compact interval. His approach is based on the investigation of the behavior of u at the boundary of the interval via the maximum principle. The main difficulty in our case is that a circle has no boundary, so that his argument does not directly apply. This difficulty is overcome by comparing u with its reflection (Lemma 1 in § 2).

§ 2. **Proof of Theorem.** The following lemma is essential:

Lemma 1. *If $\phi\in\omega(u)$ and $\phi'(a)>0$ (resp. <0) at some $a\in\boldsymbol{R}$, then there exists $T>0$ such that $u_x(a,t)\geq 0$ (resp. ≤ 0) for any $t\geq T$.*

Proof. Let

$$\xi(x)=\phi(x)-\phi(2a-x),$$
$$v(x,t)=u(x,t)-u(2a-x,t).$$

By (1.1) and (1.3), we obtain

$$(2.1)\qquad \begin{cases} \xi''+b(x)\xi=0, & x\in\boldsymbol{R},\\ \xi(a)=\xi(a+1)=0, & \\ \xi'(a)=2\phi'(a)>0, & \end{cases}$$

$$(2.2)\qquad \begin{cases} v_t=v_{xx}+b(x,t)v, & x\in\boldsymbol{R},\ t>0,\\ v(a,t)=v(a+1,t)=0, & t\geq 0,\\ v_x(a,t)=2u_x(a,t), & t>0, \end{cases}$$

where $b(x)$ and $b(x,t)$ are continuous functions. We remark that $\xi\in\omega(v)$, where $\omega(v)$ is defined in the same manner as in (1.2). Combining (2.1) and (2.2), and following the argument found in [3], one can prove Lemma 1. We omit the details.

Using Lemma 1, one can easily prove the following:

Lemma 2. *Let $\phi,\psi\in\omega(u)$ be both nonconstant and let $a\in\boldsymbol{R}$. Then $\phi'(a)>0$ (resp. <0, $=0$) implies $\psi'(a)>0$ (resp. <0, $=0$).*

Proof of Theorem. We consider only the case where $\omega(u)$ contains a nonconstant function, say ψ, since otherwise the proof is much easier by the connectedness of $\omega(u)$ and the maximum principle. By making a translation in x-axis if necessary, we may assume without loss of generality that $\psi'(0)=0$. Let

$$I(u)=\{\lambda\in\boldsymbol{R}:\phi(\cdot\,;\lambda)\in\omega(u)\}.$$

Then, by Lemma 2, $\omega(u)=\{\phi(\cdot\,;\lambda):\lambda\in I(u)\}$. As $\omega(u)$ is connected and closed, $I(u)$ is a closed interval containing $\psi(0)$. Since $\Lambda(f)$ contains no interior point and $\phi(\cdot\,;\lambda)$ are nonconstant for $\lambda\in\boldsymbol{R}$ sufficiently near $\psi(0)$, it follows that $\omega(u)$ has only one element. This proves the theorem.

Remark. Our assumption on $\Lambda(f)$ in Theorem is very technical and it does not cover the linear case, say $f(u)=4\pi^2k^2u$, with k a nonzero integer, although in this case the uniqueness of the ω-limit point is obvious. But we note that it covers a wide class of nonlinear equations including the case where $f(u)=A|u|^p u$ $(A>0, p>0)$ and the case where $f(u)=(A-B|u|^p)u$ $(A>0,\ B>0,\ p>0)$.

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