# 69. On the Existence and Asymptotic Behavior of Solutions of Nonlinear Heat Flow with Memory

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**1. Introduction and result.** We shall consider the problem of nonlinear heat flow in materials with memory:

$$(\mathrm{M})\begin{cases} \dfrac{\partial}{\partial t}\left[u(t,x)+\displaystyle\int_{-\infty}^{t}k(t-s)u(s,x)ds\right]=\sigma(u_x(t,x))_x+h(t,x), \\ \qquad\qquad t\in \boldsymbol{R}^+,\ x\in(0,1), \\ u_x(t,0)\in\beta_0(u(t,0)),\quad -u_x(t,1)\in\beta_1(u(t,1)),\quad t\in\boldsymbol{R}, \\ u(t,x)=u_0(x),\quad t\in(-\infty,0],\quad x\in(0,1). \end{cases}$$

Throughout, $k$, $\sigma$ and $\beta_i$ $(i=0,1)$ are always assumed that

($k$) $k\in L^1(0,\infty)$, nonnegative, nonincreasing and bounded.

($\sigma$) $\sigma\in C^1(\boldsymbol{R})$, $\sigma(0)=0$, $\sigma(\boldsymbol{R})=\boldsymbol{R}$, and $\sigma$ is strictly increasing.

($\beta$) $\beta_i(i=0,1)$ are maximal monotone graphs in $\boldsymbol{R}\times\boldsymbol{R}$ satisfying $0\in\beta_i(0)$.

Our purpose is to obtain the following

**Theorem 1.1.** *Let $h\in L^1(0,\infty;L^p(0,1))$ and $u_0\in L^p(0,1)$, $1<p<\infty$. Assume that the one of the following conditions is satisfied:*

(A) *$\beta_i\equiv 0$ for $i=0$ and 1.*

(B) *$\sigma$ satisfies $\sigma'>0$ and*

$$(1.1)\qquad \int_0^\infty r\cdot\min\{\sigma'(s):|s|\le r\}dr=\infty,\quad \textit{in addition to } (\sigma),$$

*and $\beta_i$ satisfies*

$$(1.2)\qquad \sup\{|y|:y\in R(\beta_i)\}<\infty \quad \textit{for } i=0 \textit{ or } 1\ (R \textit{ means a range}).$$

(C) *$\sigma$ satisfies*

$$(1.3)\qquad \exists\delta>0:\sigma'\ge\delta,\quad \textit{in addition to } (\sigma).$$

*Then the unique "generalized solution" $u(t,x)$ of* (M) *(defined below) exists and it converges strongly in $L^p(0,1)$ to some constant $\zeta_\infty$ satisfying $0\in\beta_i(\zeta_\infty)$ $(i=0,1)$ as $t\to\infty$.*

**Remarks.** 1) The condition (1.1) was introduced by [11] and it states roughly that the gradient of $\sigma$ is allowed to lie to some extent. Note that (1.3) implies (1.1).

2) In the case of (A), it is easy to see that

$$\zeta_\infty=\int_0^1 u_0(x)dx+\left(1+\int_0^\infty k(s)ds\right)^{-1}\int_0^\infty\int_0^1 h(t,x)dxdt \qquad (\text{cf. } [1]).$$

3) In the case of Dirichlet boundary condition, if (1.3) is assumed, we can obtain the estimate of decay corresponding to an exponential decay ([3], [7]):

$$(1.4)\qquad \|u(t)\|_p\le\left(\int_t^\infty r(\tau)d\tau\right)\|u_0\|_p+\omega^{-1}\int_0^t r(t-\tau)[u(\tau),h(\tau)]_+d\tau,$$

where $\omega>0$ is some constant and $r$ is defined by $r+\omega b*r=\omega b$, $b+k*b=1$, and $[x, y]_+=\lim_{\lambda\downarrow 0}(\|x+\lambda y\|_p-\|x\|_p)/\lambda$, $\|\cdot\|_p$ is the $L^p$-norm.

**2. Reduction to the abstract equation.** Let $1<p<\infty$. Define $A$ by

$$D(A)=\{u\in C^1[0,1]: u'(0)\in\beta_0(u(0)),\ -u'(1)\in\beta_1(u(1)),\ \text{and } \sigma(u')\in W^{1,p}(0,1)\}$$
$$Au=-\sigma(u')' \quad \text{for } u\in D(A).$$

With this $A$, $(M)$ can be interpreted as an abstract equation in $L^p(0,1)$:

$$(E)\begin{cases}(d/dt)u(t)+Au(t)+G(u)(t)\ni h(t)+k(t)u_0, & t\in \boldsymbol{R}^+,\\ u(0)=u_0,\end{cases}$$

where $G(u)(t)=k(0)u(t)+\int_0^t u(t-s)dk(s)$. A function $u\in C(\boldsymbol{R}^+;\overline{D(A)})$ is called simply a solution of (E) if it is an "integral solution" of $(E)$ considering $h(t)+k(t)u_0-G(u)(t)$ as an inhomogeneous term ([4]). Then we define the "generalized solution" of (M) by $u(t,x)=[u(t)](x)$, where $u(t)$ is the solution of (E).

To obtain Theorem 1.1, we will apply the following abstract results concerning (E):

**Theorem 2.1** ([4, 6, 10]). *Let $h\in L^1(0,\infty;X)$ and $u_0\in\overline{D(A)}$. Assume that $A$ is m-accretive, $A^{-1}0\neq\varnothing$, and $A$ satisfies the convergence condition (see below). If $(I+A)^{-1}$ is compact, then the unique solution $u(t)$ of $(E)$ exists and converges strongly to an element of $A^{-1}0$ as $t\to\infty$.*

If $A$ is $m$-accretive in $X=L^p(0,1)$ and $A^{-1}0\neq\varnothing$, the nearest point mapping $P$ onto $A^{-1}0$ is well-defined and continuous since $A^{-1}0$ is a closed convex subset of $L^p(0,1)$. Denote by $J$ the single-valued duality mapping in $X$. For the definition of the convergence condition, we refer to [9] and here we recall the sufficient condition for $A$ to satisfy it:

**Proposition 2.2** ([9]). *Let $A$ be m-accretive with $A^{-1}0\neq\varnothing$. If $\langle y, J(x-Px)\rangle>0$ for every $[x,y]\in A$ with $x\notin A^{-1}0$, and the resolvent $(I+A)^{-1}$ is compact, then $A$ satisfies the convergence condition.*

Now, we have only to prove that:

**Proposition 2.3.** *Assume that the one of the conditions* (A), (B) *and* (C) *is satisfied. Then $A$ is m-accretive in $L^p(0,1)$, the resolvent $(I+A)^{-1}$ is a compact operator, and $A$ satisfies the convergence condition.*

**3. Sketch of proof of Proposition 2.3.** It is easy to see that $A$ is accretive in $L^p(0,1)$ from the form of tangent function $[\cdot,\cdot]_+$ in $L^p(0,1)$. In the case of (A), we make use of the results of [z] and obtain $W^{1,1}(0,1)\subset R(I+A)$, whereas in the cases of (B) and (C), we have $C[0,1]\subset R(I+A)$ by [11]. Therefore in order to show that $A$ is $m$-accretive, it suffices to show that $A$ is closed in $L^p(0,1)$. Let $u_n\in D(A)$ be such that $u_n\to u$ and $-\sigma(u_n')'\to v$ in $L^p(0,1)$. In the cases of $(A)$ and $(B)$, it follows from

$$\sigma(u_n'(x))-\sigma(u_n'(0))=\int_0^x \sigma(u_n'(\tau))'d\tau \tag{3.1}$$
$$\left(\text{or}\quad \sigma(u_n'(1))-\sigma(u_n'(x))=\int_x^1 \sigma(u_n'(\tau))'d\tau\right)$$

that $\|u_n'(x)\|\leq C$. (Hereafter $C$ denotes a universal constant.) From this, we can get $u \in W^{2,p}(0,1) \subset C^1[0,1]$, $\sigma(u') \in W^{1,p}(0,1)$, and $v = -\sigma(u')'$. In the case of (A), the desired boundary condition of $u$ is easily checked. On the other hand in the case of (B), we further show the estimate $\|u_n''\|_p \le C$, so that

(3.2) $$\|u_n\|_{W^{2,p}(0,1)} \le C.$$

Then we obtain $u_n \to u$ in $W^{2,p}(0,1) \subset C^1[0,1]$, and by the closedness of $\beta_i$ $(i=0,1)$, the boundary condition of $u$ is satisfied, and so $A$ is closed.

In the case of (C), it follows from

$$\int_0^1 \delta^p |u_n''|^p \le \int_0^1 |\sigma'(u_n')u_n''|^p = \int_0^1 |\sigma(u_n')'|^p$$

that $\|u_n''\|_p \le C/\delta$. Then since

(3.3) $$\|u_n'\|_p^p \le K(\|u_n''\|_p^p + \|u_n\|_p^p), \quad K \text{ depends only on } p,$$

we have the estimate (3.2) and hence $A$ is closed as shown above.

To prove the compactness of $(I+A)^{-1}$, let $f \in L^p(0,1)$ and take $u \in D(A)$ such that $u + Au = f$. In the cases of (A) and (B), by the equation (3.1) with $u$ in place of $u_n$ and the accretivity of $A$ together with $0 \in A0$, the estimate $\|u\|_{W^{1,p}} \le C + C\|f\|_p$ holds. In the case of (C), we have $\|u''\|_p \le (C/\delta)(\|u\|_p + \|f\|_p) \le (2C/\delta)\|f\|_p$ and by the inequality like (3.3), the estimate $\|u\|_{W^{1,p}} \le C\|f\|_p$ follows. Since the embedding $W^{1,p}(0,1) \subset L^p(0,1)$ is compact, we conclude that $(1+A)^{-1}$ is compact.

Finally, noting that $u$ is not constant if $u \in D(A) \backslash A^{-1}0$, it is not difficult to see that

$$\langle Ax, J(x-Px)\rangle > 0 \qquad \text{for any } x \in D(A) \text{ with } x \notin A^{-1}0.$$

Thus by Proposition 2.2, $A$ satisfies the convergence condition. Q.E.D.

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