5. *Characterization of the Eigenfunctions in the Singularly Perturbed Domain. II*

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In this paper, we give some elaborate estimates concerning the eigenfunction which behaves singularly when the domain is singularly perturbed. J.T. Beale [1] has characterized the set of scattering frequencies (i.e. the square root of the spectrum) of the exterior domain of a bounded obstacle with a partially open cavity when the channel to the cavity is very narrow. In our previous works [2] and [3], we have dealt with a Dumbell type domain; $\Omega(\zeta)=D_1\cup D_2\cup Q(\zeta)$ where $Q(\zeta)$ approaches a line segment as $\zeta\to 0$ (which is a similar domain perturbation to that of J. T. Beale) and we have characterized the eigenfunctions of the operator $-\Delta$ in the case of the Neumann boundary condition. Roughly speaking, the complete system of the eigenvalues $\{\mu_k(\zeta)\}_{k=1}^{\infty}$ and the eigenfunctions $\{\Phi_{k,\zeta}\}_{k=1}^{\infty}$ orthonormalized in $L^2(\Omega(\zeta))$ are separated as follows,

$$\{\mu_k(\zeta)\}_{k=1}^{\infty}=\{\omega_k(\zeta)\}_{k=1}^{\infty}\cup\{\lambda_k(\zeta)\}_{k=1}^{\infty}$$
$$\{\Phi_{k,\zeta}\}_{k=1}^{\infty}=\{\phi_{k,\zeta}\}_{k=1}^{\infty}\cup\{\psi_{k,\zeta}\}_{k=1}^{\infty}$$

where

$$\lim_{\zeta\to 0}\|\phi_{k,\zeta}\|_{L^2(Q(\zeta))}=0,\qquad \lim_{\zeta\to 0}\|\psi_{k,\zeta}\|_{L^2(D_1\cup D_2)}=0,$$
$$\limsup_{\zeta\to 0}\|\phi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))}<+\infty,\quad \lim_{\zeta\to 0}\|\psi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))}=+\infty.$$

More precisely, $\phi_{k,\zeta}$ approaches the k-th eigenfunction on $D_1\cup D_2$ uniformly and $\psi_{k,\zeta}$ approaches the k-th eigenfunction

$$\frac{1}{d_{n-1}^{1/2}\zeta^{(n-1)/2}}\sin\frac{1}{2}k\pi(x_1+1)$$

of $-d^2/dx_1^2$ on the line segment $L=\bigcap_{\zeta>0}\overline{Q(\zeta)}$ with the Dirichlet boundary condition on the endpoints of L in some sense. The asymptotic behavior of $\phi_{k,\zeta}$ when $\zeta\to 0$ has been obtained globally in $\Omega(\zeta)$ in [2]. In this paper we obtain the exact decay estimate of $\psi_{k,\zeta}$ in $D_1\cup D_2$ when $\zeta\to 0$. The estimates or methods obtained are very useful when we deal with a construction of the solutions of some semilinear elliptic equation on the singularly perturbed domain.

§1. **Formulation.** We specify the singularly perturbed domain $\Omega(\zeta)$ in $\boldsymbol{R}^n$ in the following form,

$$\Omega(\zeta)=D_1\cup D_2\cup Q(\zeta)$$

where D_i $(i=1, 2)$ and $Q(\zeta)$ are defined in the following conditions where $x'=(x_2, x_3, \cdots, x_n)\in\boldsymbol{R}^{n-1}$.

(A.1) D_1 and D_2 are bounded domains in $\boldsymbol{R}^n$ (mutually disjoint) with smooth boundaries which satisfy the following conditions for some positive constant $\zeta_* > 0$.

$$\bar{D}_1 \cap \{x=(x_1, x') \in \boldsymbol{R}^n | x_1 \leqq 1, |x'| < 3\zeta_*\} = \{(1, x') \in \boldsymbol{R}^n | |x'| < 3\zeta_*\}$$

$$\bar{D}_2 \cap \{x=(x_1, x') \in \boldsymbol{R}^n | x_1 \geqq -1, |x'| < 3\zeta_*\} = \{(-1, x') \in \boldsymbol{R}^n || x'| < 3\zeta_*\}$$

(A.2) $Q(\zeta) = R_1(\zeta) \cup R_2(\zeta) \cup \Gamma(\zeta)$

$R_1(\zeta) = \{(x_1, x') \in \boldsymbol{R}^n | 1-2\zeta < x_1 \leqq 1, |x'| < \zeta\rho((x_1-1)/\zeta)\}$

$R_2(\zeta) = \{(x_1, x') \in \boldsymbol{R}^n | -1 \leqq x_1 < -1+2\zeta, |x'| < \zeta\rho((-1-x_1)/\zeta)\}$

$\Gamma(\zeta) = \{(x_1, x') \in \boldsymbol{R}^n | -1+2\zeta \leqq x_1 \leqq 1-2\zeta, |x'| < \zeta\}$

where $\rho \in C^0((-2, 0]) \cap C^\infty((-2, 0))$ is a positive valued monotone increasing function such that $\rho(0)=2$, $\rho(s)=1$ for $s \in (-2, -1)$ and the inverse function ρ^{-1}: $[1, 2] \longrightarrow [-1, 0]$ satisfies $\lim_{\xi \to 2} (d^k \rho^{-1}/d\xi^k)(\xi)=0$ for any nonnegative integer k. Hereafter we denote the points $p_1=(1, 0, \cdots, 0)$, $p_2=(-1, 0, \cdots, 0)$ and the sets $L = \cap_{0<\zeta<\zeta_*} \overline{Q(\zeta)} = \{(z, 0, \cdots, 0) \in \boldsymbol{R}^n | -1 \leqq z \leqq 1\}$, $\Sigma_i(\eta) = \{x \in D_i | |x-p_i| < \eta\}$ for $\eta > 0$ and $i=1, 2$.

Let $\{\mu_k(\zeta)\}_{k=1}^\infty$ be the complete system of the eigenvalues of (1.1) arranged in increasing order (counting multiplicity).

$$\text{(1.1)} \qquad \begin{cases} \Delta\Phi + \mu\Phi = 0 & \text{in } \Omega(\zeta), \\ \dfrac{\partial\Phi}{\partial\nu} = 0 & \text{on } \partial\Omega(\zeta), \end{cases}$$

where $\Delta = \sum_{k=1}^n \partial^2/\partial x_k^2$ and ν denotes the unit outward normal vector on $\partial\Omega(\zeta)$.

Let $\{\omega_k\}_{k=1}^\infty$ and $\{\phi_k\}_{k=1}^\infty$ be respectively the sequence of the eigenvalues arranged in increasing order and the complete system of the corresponding orthonormalized eigenfunctions of the following eigenvalue problem in $D_1 \cup D_2$.

$$\text{(1.2)} \qquad \begin{cases} \Delta\phi + \omega\phi = 0 & \text{in } D_1 \cup D_2, \\ \dfrac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial D_1 \cup \partial D_2. \end{cases}$$

$(0=\omega_1=\omega_2 \leqq \omega_3 \leqq \cdots \to \infty,\ (\phi_k \phi_j)_{L^2(D_1 \cup D_2)} = \delta_{k,j},\ k, j \geqq 1)$.

We put $\lambda_k = (k\pi/2)^2$ and $S_k(z) = \sin(k\pi/2)(z+1)$ $(k \geqq 1)$ which are respectively the eigenvalues and the eigenfunctions of the operator $-d^2/dz^2$ on the line segment L with the Dirichlet boundary condition on the endpoints of L.

We also assume the following condition

$$\text{(A.3)} \qquad \{\lambda_k\}_{k=1}^\infty \cap \{\omega_k\}_{k=1}^\infty = \emptyset.$$

By applying the method of J. T. Beale [1], we can separate the set of the eigenvalues of (1.1) for small $\zeta > 0$, i.e. $\{\mu_k(\zeta)\}_{k=1}^\infty$ is expressed as follows

$$\text{(1.3)} \qquad \{\mu_k(\zeta)\}_{k=1}^\infty = \{\omega_k(\zeta)\}_{k=1}^\infty \cup \{\lambda_k(\zeta)\}_{k=1}^\infty,$$

where $\lim_{\zeta \to 0} \omega_k(\zeta) = \omega_k$, $\lim_{\zeta \to 0} \lambda_k(\zeta) = \lambda_k$ $(k=1, 2, 3, \cdots)$. By [2], we can choose a complete system of the eigenfunctions $\{\Phi_{k,\zeta}\}_{k=1}^\infty$ of (1.1) which are decomposed below according to the decomposition of the eigenvalues (1.3),

$$\{\Phi_{k,\zeta}\}_{k=1}^{\infty}=\{\phi_{k,\zeta}\}_{k=1}^{\infty}\cup\{\psi_{k,\zeta}\}_{k=1}^{\infty},$$
$$(\Phi_{k,\zeta}\cdot\Phi_{j,\zeta})_{L^2(\Omega(\zeta))}=\delta_{k,j}\quad(k,\ j\geqq 1),$$

where $\phi_{k,\zeta}$ converges to ϕ_k uniformly in $D_1\cup D_2$ and $\phi_{k,\zeta}|_{Q(\zeta)}$ is uniformly approximated by some solution of the boundary value problem of the ordinary differential equation on L and $d_{n-1}^{1/2}\zeta^{(n-1)/2}\psi_{k,\zeta}$ converges to 0 uniformly in $D_1\cup D_2$ and $d_{n-1}^{1/2}\zeta^{(n-1)/2}\psi_{k,\zeta}|_{Q(\zeta)}$ is uniformly approximated by $S_k(x_1)$. (See [2].) But this characterization does not contain the behavior in $D_1\cup D_2$ of $\psi_{k,\zeta}$ in the sense of uniform convergence. In this note we give the decay rate or behavior of $\psi_{k,\zeta}$ itself in $D_1\cup D_2$ where d_{n-1} is the $(n-1)$-dimensional Lebesgue measure of the unit ball of in $\boldsymbol{R}^{n-1}$.

§ 2. Main results.

Theorem. *Assume $n\geqq 3$. Then, there exists a positive constant $\eta_*>0$ such that,*

$$0<\liminf_{\zeta\to 0}\ \inf_{x\in R_i(\zeta)\cup\Sigma_i(2\zeta)}\ \zeta^{(n-3)/2}|\psi_{k,\zeta}(x)| \leqq\limsup_{\zeta\to 0}\ \sup_{x\in R_i(\zeta)\cup\Sigma_i(2\zeta)}\ \zeta^{(n-3)/2}|\psi_{k,\zeta}(x)|<+\infty, \tag{2.1}$$

$$0<\liminf_{\zeta\to 0}\ \inf_{x\in \Sigma_i(\eta)\setminus\Sigma_i(2\zeta)}\ \zeta^{-(n-1)/2}|x-p_i|^{n-2}|\psi_{k,\zeta}(x)| \leqq\limsup_{\zeta\to 0}\ \sup_{x\in \Sigma_i(\eta)\setminus\Sigma_i(2\zeta)}\ \zeta^{-(n-1)/2}|x-p_i|^{n-2}|\psi_{k,\zeta}(x)|<+\infty, \tag{2.2}$$

$$0<\liminf_{\zeta\to 0}\ \sup_{x\in D_i\setminus\Sigma_i(\eta)}\ \zeta^{-(n-1)/2}|\psi_{k,\zeta}(x)| \leqq\limsup_{\zeta\to 0}\ \sup_{x\in D_i\setminus\Sigma_i(\eta)}\ \zeta^{-(n-1)/2}|\psi_{k,\zeta}(x)|<+\infty, \tag{2.3}$$

$$0<\liminf_{\zeta\to 0}\zeta^{-(n-1)/2}\|\psi_{k,\zeta}\|_{L^1(\Omega(\zeta))} \leqq\limsup_{\zeta\to 0}\zeta^{-(n-1)/2}\|\psi_{k,\zeta}\|_{L^1(\Omega(\zeta))}<+\infty, \tag{2.4}$$

for any $k\geqq 1$, $\eta\in(0,\eta_*)$ and $i=1, 2$.

Remark that $\lim_{\zeta\to 0}\|\psi_{k,\zeta}\|_{L^1(\Omega(\zeta))}=0$ holds while $\|\psi_{k,\zeta}\|_{L^2(\Omega(\zeta))}=1$.

Corollary. *There exist positive constants $\zeta_0(k)$, $c_1(k)$, $c_2(k)$ for any $k\geqq 1$ such that*

$$\frac{c_1(k)\zeta^{(n-1)/2}}{|x-p_i|^{n-2}}\leqq|\psi_{k,\zeta}(x)|\leqq\frac{c_2(k)\zeta^{(n-1)/2}}{|x-p_i|^{n-2}}\qquad(x\in\Sigma_i(\eta_*)\setminus\Sigma_i(2\zeta))$$

holds for $\zeta\in(0,\zeta_0(k))$ and $i=1, 2$.

The details will appear elsewhere.

References

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