# 10. *Lifting of Local Subdifferentiations and Elliptic Boundary Value Problems on Symmetric Domains.* II

By Takashi SUZUKI*) and Ken'ichi NAGASAKI**)

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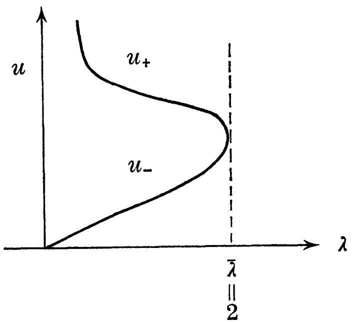
Our purpose is to study nonlinear eigenvalue problem

(1) $$-\Delta u=\lambda e^{u} \quad (\text{in } \Omega), \quad u=0 \quad (\text{on } \partial\Omega)$$

for $\lambda>0$ on $\Omega=\{x \mid a<|x|<1\}\subset \boldsymbol{R}^2$, where $0<a<1$. From variational method, we shall show the existence of multiple non-radial solutions for (1). Namely, we seek the solutions by lifting of local subdifferentiations developed in [6], and then separate critical values by Steiner's symmetrization according to the argument by Kawohl [3]. Meanwhile we shall make use of radial solutions for (1) on the ball. Thus, our plan is; i. Description of the solutions for (1) on $\Omega_0=\{|x|<1\}$, ii. Description of radial solutions for (1) on $\Omega_a=\{a<|x|<1\}$ $(0<a<1)$ and iii. Existence of non-radial solutions for (1) on $\Omega_a$.

We note that the equation $-\Delta u=\lambda e^{u}$ (in $\Omega$) has an integral (Liouville [4]). Thus it is equivalent to $(\lambda/8)^{1/2}e^{u/2}=\rho(F)=|F'|/(1+|F|^2)$, where $F$ is a meromorphic function on $\Omega$ such that $\rho(F)>0$. Therefore, (1) is nothing but to find $F$ such that $\rho(F)|_{\partial\Omega}=(\lambda/8)^{1/2}$.

*Solutions of* (1) *for* $\Omega_0=\{|x|<1\}\subset \boldsymbol{R}^2$: Every solution $u=u(x)$ of (1) is positive so that is radial in this case (Gidas-Ni-Nirenberg [2]). Hence the result of Gel'fand [1] supplies a complete diagram of the solutions of (1). In terms of the Liouville integral given above, they are given through $F(z)=Cz$ with a $C>0$ satisfying $\rho(F)|_{\partial\Omega}=C/(1+C^2)=(\lambda/8)^{1/2}$. Hence for $\lambda>2$ (1) has no solution. According to $\lambda=2$ and $0<\lambda<2$, (1) has exactly one and two solutions $u=u_{\pm}$. They are described through the parameter $\kappa=1/C^2$, which is given as $\kappa^{1/2}=\kappa_{\pm}^{1/2}=(2/\lambda)^{1/2}(1\mp\sqrt{1-\lambda/2})$ for $0<\lambda\leq 2$. That is, $(\lambda/8)^{1/2}e^{u_{\pm}/2}=\kappa_{\pm}^{1/2}/(|x|^2+\kappa_{\pm})$. See the figure given below.




*) Department of Mathematics, Faculty of Science, University of Tokyo.
**) Department of Mathematics, Faculty of Engineering, Chiba Institute of Technology.

Hence $u^+$ makes one-point blow-up, i.e., $u^+(x)\sim 4\log(1/|x|)$ as $\lambda\downarrow 0$. We now recall the geometrical meaning of $\rho(F)$ for the meromorphic function $w=F(z)$, which was first noted in Nagasaki-Suzuki [5]. Namely, let $K$ be the Riemann sphere with unit diameter and tangent to $w$-plane at the origin. The mapping $F:\Omega\to \boldsymbol{C}\cup\{\infty\}$ may be regarded as $\bar{F}:\Omega\to K$. Then we have $\rho(F)=d\omega/|dz|$, where $d\omega$ and $|dz|$ denote the line elements on $K$ and $\Omega$, respectively, the former being induced by the latter through $\bar{F}$. Hence $S_1=\int_\Omega \rho(F)^2\,dx=\frac{\lambda}{8}\int_\Omega e^u dx$ denotes the area of $\bar{F}(\Omega)$ on $K$. In view of this fact, we can easily see that every solution $g={}^T(u,\lambda)$ of (1) is parametrized by $S=\lambda\int_\Omega e^u dx\in(0,8\pi)$ when $\Omega=\Omega_0=\{|x|<1\}\subset\boldsymbol{R}^2$. Through an elementary calculation we have

$$\mu_0(S)\equiv\int_\Omega e^u dx\left(=\frac{S}{\lambda}\right)=8\pi^2/(8\pi-S)\{1+o(1)\}\quad\text{and}\tag{2}$$

$$\int_\Omega|\nabla u|^2dx=16\pi\left(\log\frac{1}{8\pi-S}\right)\{1+o(1)\}\qquad\text{as } S\nearrow 8\pi.$$

*Radial solutions of* (1) *for* $\Omega_a=\{a<|x|<1\}\subset\boldsymbol{R}^2$ $(0<a<1)$: Writing (1) in polar coordinate to integrate it, we can give all radial solutions for (1) explicitly in this case. In use of the Liouville integral, these are realized as $F(z)=\beta^{1/2}z^\alpha$ $(\alpha,\beta>0)$, where $\beta$ and $\alpha$ are determined through $\rho(F)|_{|z|=a,1}=(\lambda/8)^{1/2}$. Consequently, we obtain the same diagram as Fig. 1 for radial solutions of (1) on $\Omega=\Omega_a$. However, in this case $u_+$ makes the entire blow-up: $u_+(x)\to\infty$ $(x\in\Omega)$ as $\lambda\downarrow 0$. Further, $P_+=\lambda e^{u_+}$ tends to $+\infty$ and 0, according to $|x|=\sqrt{a}$ and $|x|\neq\sqrt{a}$, respectively. Every solution $g={}^T(u,\lambda)$ of (1) for $\Omega=\Omega_a$ is parametrized by $S=\lambda/\alpha\int_\Omega e^u dx\in(0,8\pi)$, $\alpha$ being determined as before by $\lambda$. Actually, $S$ denotes the area in $K$ of the image of $\Omega$ under $\bar{F}$. Note that in this case $F$ is $\alpha$-fold. We have

$$\mu_a(S)=\int_\Omega e^u dx\left(=\frac{\alpha S}{\lambda}\right)=8\pi^2(a+a^{-1})\frac{a\log a}{(8\pi-S)\log(8\pi-S)}\{1+o(1)\}\tag{3}$$

$$\text{and}\quad\int_\Omega|\nabla u|^2dx=\frac{8\pi}{\log(1/a)}\left(\log\frac{1}{8\pi-S}\right)^2\{1+o(1)\}\qquad\text{as } S\nearrow 8\pi.$$

These calculations are never trivial but rather elementary.

*Existence of non-radial solutions of* (1) *for* $\Omega_a=\{a<|x|<1\}\subset\boldsymbol{R}^2$ $(0<a<1)$: We can seek non-radial solutions by variational method. One tool is Lagrange multiplier and the other is lifting of local subdifferentiations. Namely, let $T_k$ be the rotation operator of independent variables described in [6] for $k=1,2,\cdots$.

Setting $K_\infty(\mu)=\left\{v\in H_0^1(\Omega_a)|v \text{ is radial and } \int_\Omega e^v dx=\mu\right\}$ and $K_k(\mu)=\left\{v\in H_0^1(\Omega_a)|T_kv=v,\ \int_\Omega e^v dx=\mu\right\}$ $(k=1,2,\cdots)$ for $\mu>|\Omega|$, we consider the variational problem

$$j_k(\mu)=\operatorname{Inf}\left\{\frac{1}{2}\int_\Omega|\nabla v|^2dx\mid v\in K_k(\mu)\right\}\qquad(k=1,2,\cdots,\infty).\tag{4}$$

From Trudinger's inequality compactness of the mapping

$$v \in H_0^1(\Omega_a) \longmapsto \int_\Omega e^v dx \in \boldsymbol{R}$$

follows, so that the minimizers $u_k \in K_k(\mu)$ of (4) exist. The function $u=u_k$ satisfies $\lambda e^u \in \partial(\varphi+1_K)(u)$ for $K=K_k$ with a Lagrange multiplier $\lambda \in \boldsymbol{R}$, where $\varphi(v)=1/2\int_\Omega |\nabla v|^2 dx$ $(v \in H_0^1(\Omega))$. By virtue of $u \in K$ $(=K_k)$, the invariance of $f=\lambda e^u$ with respect to $\partial\varphi$ in $K$ holds. Hence we obtain $\lambda e^u \in \partial\varphi(u)$ and $u \in \boldsymbol{K}$, which means that $u$ solves (1), $T_k u=u$ and $\int_\Omega e^u\, dx=\mu$. In case $\lambda \leqq 0$, $u \leqq 0$ holds and hence $\mu=\int_\Omega e^u dx \leqq |\Omega|$. Thus $\mu>|\Omega|$ implies $\lambda>0$. See our forthcoming paper for more general form of the Lagrange multiplier principle.

We now claim

(5) $\quad m|n$ $(m \neq n)$ implies $j_m(\mu)<j_n(\mu)$ provided that $j_n(\mu)<j_\infty(\mu)$

and

(6) $\quad j_k(\mu)<j_\infty(\mu)$ for each $k=1, 2, \cdots$, when $\mu \nearrow +\infty$,

which guarantees

**Theorem.** *For each positive integer $k$, there exists a family of solutions $g={}^T(u, \lambda)$ of (1) for $\Omega=\Omega_a$, whose modes are $k$ and $\int_\Omega e^u dx=\mu\nearrow\infty$.*

*Outline of proof of* (5): We can apply the argument by Kawohl [3]. Let $u_n^*$ $(re^{i\theta})$ be the Steiner symmetrization of $u_n$ on

$$D_m=\left\{-\frac{\pi}{m}<\theta<\frac{\pi}{m},\ a<r<1\right\}.$$

Then, $\mu=\int_\Omega e^{u_n} dx=\int_\Omega e^{u_n^*} dx$ and $T_n u_n^* \not\equiv u_n^*$ because of $u_n \notin K_\infty(\mu)$. Hence $u_n^* \in K_m(\mu)\setminus K_n(\mu)$. On the other hand $u_n \in K_n(\mu)$ so that $u_n^* \not\equiv u_n$ (modulo rotation of independent variables $x$) and hence $\int_\Omega |\nabla u_n|^2 dx>\int_\Omega |\nabla u_n^*|^2 dx$. Thus, we have the conclusion.

*Outline of proof of* (6): The mapping $S \mapsto \mu_a(S)$ is one-to-one as $S\nearrow 8\pi$, $\mu_a(S)$ being defined in (3). Hence

$$j_\infty(\mu_a(S))=\frac{4\pi}{\log(1/a)}\left(\log\frac{1}{8\pi-S}\right)^2\{1+o(1)\} \qquad \text{as } S\nearrow 8\pi. \tag{7}$$

To estimate $j_k(\mu)$ from above, we take a ball $\omega_k$ in $D_k$ whose center and radius are denoted by $x_0$ and $\varepsilon$, respectively. Through the (radial) solution ${}^T(\Phi, \lambda)={}^T(\Phi(t), \lambda(t))$ of (1) for $\Omega=\Omega_0$ with $\lambda\int_\Omega e^\Phi dx=t \in (0, 8\pi)$, we set $\varphi(x)=\Phi((1/\varepsilon)(x-x_0))$ for $x \in \omega_k$ and $\Phi=0$ for $x \in \Omega_a\setminus\omega_k$. Translating $\theta$ in $\varphi(re^{i\theta})$ by $2\pi/k$ we take $k$-functions $\varphi_1, \cdots, \varphi_k$ and put $v=v(t)=\varphi_1+\cdots+\varphi_k$. At this point we specify the parameter $t \in (0, 8\pi)$ so that $\int_{\Omega_a} e^v ds=\mu_a(S)$ for given $S \in (0, 8\pi)$. As $S\nearrow 8\pi$, $t=t(S)\nearrow 8\pi$ follows. Then, we can show that $\int_{\Omega_a}|\nabla v|^2 dx=16\pi k \log(1/8\pi-S)\{1+o(1)\}$ as $S\nearrow 8\pi$. Thus,

$$(8) \qquad j_k(\mu_a(S)) \leqq 16\pi k \log(1/8\pi - S)\{1+o(1)\} \qquad \text{as } S \nearrow 8\pi.$$

The relations (7) and (8) imply (6).

**Remark.** It is interesting whether the non-radial solutions bifurcate from radial ones or not. By the theory of [7], the problem is reduced to studying the degeneracy of linearized operators around radial solutions. However, the linearized eigenvalue problem can be transformed into that on associated Legendre equation. Thus, we can discuss the bifurcation problem through the asymptotic analysis for associated Legendre equation. We shall study it in a forthcoming paper.

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