3. Sums of a Certain Class of q-series

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M. Vowe and H.-J. Seiffert [6] evaluated the sum:

$$\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{1}{2^k(n+k+1)}=\frac{2^n(n-1)!\,n!}{(2n)!}-\frac{2^{-n}}{n} \tag{1}$$
$$(n\in N=\{1,2,3,\cdots\})$$

by identifying it with an Eulerian integral. Subsequently, in our attempt in [4] to find the sum (1), *without* considering this Eulerian integral, we were led naturally to numerous interesting generalizations of (1) obtainable as useful consequences of Kummer's summation theorem [3, p. 134, Theorem 3] in the theory of the familiar (Gaussian) hypergeometric series (see [4] for details). The object of the present note is to derive certain basic (or q-) extensions of (1) and of its various generalizations given already by us [4].

For real or complex q, $|q|<1$, let

$$(\lambda;q)_0=1;\ (\lambda;q)_k=(1-\lambda)(1-\lambda q)\cdots(1-\lambda q^{k-1}),\ \forall k\in N, \tag{2}$$

and

$$(\lambda;q)_\infty=\lim_{k\to\infty}(\lambda;q)_k=\prod_{j=0}^{\infty}(1-\lambda q^j) \tag{3}$$

for an arbitrary (real or complex) parameter λ. Then a q-extension of Kummer's summation theorem [3, p. 134, Theorem 3], employed in our earlier work [4], can be written in the form (cf. [1, p. 526, Equation (1.9)]):

$$\sum_{k=0}^{\infty}q^{k(k-1)/2}\frac{(a;q)_k(q/a;q)_k}{(c;q)_k}\frac{c^k}{(q^2;q^2)_k}=\frac{(ca;q^2)_\infty(cq/a;q^2)_\infty}{(c;q)_\infty}, \tag{4}$$

or, equivalently,

$${}_2\Phi_2\left[\begin{matrix}a,\ q/a;\\ c,\ -q;\end{matrix}\ q,\ -c\right]=\frac{(ca;q^2)_\infty(cq/a;q^2)_\infty}{(c;q)_\infty} \tag{5}$$

in terms of a basic (or q-) hypergeometric ${}_r\Phi_s$ function (cf., e.g., [5, p. 347, Equation (272)]).

Defining the basic (or q-) binomial coefficient by

$$\begin{bmatrix}\lambda\\0\end{bmatrix}=1;\quad \begin{bmatrix}\lambda\\k\end{bmatrix}=(-1)^kq^{k(2\lambda-k+1)/2}\frac{(q^{-\lambda};q)_k}{(q;q)_k},\qquad k\in N, \tag{6}$$

it is easily verified that

$$\begin{bmatrix}\lambda+k-1\\k\end{bmatrix}=\frac{(q^\lambda;q)_k}{(q;q)_k}\qquad (k\in N_0=N\cup\{0\}) \tag{7}$$

and that

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$$(8)\qquad \lim_{q\to 1}\begin{bmatrix}\lambda\\ k\end{bmatrix}=\binom{\lambda}{k}\qquad (k\in \boldsymbol{N}_0)$$

for an arbitrary (real or complex) parameter λ.

Applying the relationship (7), it is not difficult to state the summation formula (4) or (5) in the (more relevant) form:

$$(9)\qquad S_{\lambda,\mu}^{(q)}\equiv\sum_{k=0}^{\infty}(-1)^k q^{k(k-\lambda+\mu)}\begin{bmatrix}\lambda-1\\ k\end{bmatrix}\frac{\begin{bmatrix}\lambda+k-1\\ k\end{bmatrix}}{(-q\,;q)_k\begin{bmatrix}\mu+k-1\\ k\end{bmatrix}}$$
$$=\frac{(1+q)^{1-\mu}\Gamma_p(\frac{1}{2})\Gamma_q(\mu)}{\Gamma_p((\lambda+\mu)/2)\Gamma_p((1-\lambda+\mu)/2)}\qquad (p=q^2),$$

where $\Gamma_q(z)$ denotes the basic (or q-) Gamma function defined by

$$(10)\qquad \Gamma_q(z)=\frac{(q\,;q)_\infty}{(q^z\,;q)_\infty}(1-q)^{1-z},$$

so that

$$(11)\qquad \Gamma_q(z+1)=\left(\frac{1-q^z}{1-q}\right)\Gamma_q(z),$$

$$(12)\qquad \Gamma_q(n+1)=\frac{(q\,;q)_n}{(1-q)^n}\qquad (n\in \boldsymbol{N}_0),$$

and, in terms of the familiar Gamma function,

$$(13)\qquad \lim_{q\to 1}\Gamma_q(z)=\Gamma(z).$$

Furthermore, since*) [2, p. 131, Equation (3.17)]

$$(14)\qquad \Gamma_q(2z)\Gamma_p(\tfrac{1}{2})=(1+q)^{2z-1}\Gamma_p(z)\Gamma_p(z+\tfrac{1}{2})\qquad (p=q^2),$$

the sum in (9) can easily be written in the following alternative form:

$$(15)\qquad S_{\lambda,\mu}^{(q)}=\frac{\Gamma_p(\mu/2)\Gamma_p((\mu+1)/2)}{\Gamma_p((\lambda+\mu)/2)\Gamma_p((1-\lambda+\mu)/2)}\qquad (p=q^2).$$

We now turn to the derivation of several interesting consequences of the general result (9) or (15). Indeed, for $\mu=\lambda+2l$ and $\mu=\lambda+2l+1$ ($l\in \boldsymbol{N}_0$), we find from (9) that

$$(16)\qquad \sum_{k=0}^{\infty}(-1)^k q^{k(k+2l)}\begin{bmatrix}\lambda-1\\ k\end{bmatrix}\{(-q\,;q)_k(q^{\lambda+k}\,;q)_{2l}\}^{-1}$$
$$=\frac{(1+q)^{1-\lambda}\Gamma_p(l+1)\Gamma_q(\lambda)}{(1-q)^{2l}\Gamma_p(\lambda+l)\Gamma_q(2l+1)}\qquad (l\in \boldsymbol{N}_0\,;\,p=q^2)$$

and

$$(17)\qquad \sum_{k=0}^{\infty}(-1)^k q^{k(k+2l+1)}\begin{bmatrix}\lambda-1\\ k\end{bmatrix}\{(-q\,;q)_k(q^{\lambda+k}\,;q)_{2l+1}\}^{-1}$$
$$=\frac{(1+q)^{\lambda}\Gamma_q(\lambda)\Gamma_p(\lambda+l+1)}{(1-q)^{2l+1}\Gamma_q(2\lambda+2l+1)\Gamma_p(l+1)}\qquad (l\in \boldsymbol{N}_0\,;\,p=q^2).$$

Multiplying both sides of (16) by $(1-q)q^{\lambda+2l-2}$, and subtracting the resulting equation from (17) with l replaced by $l-1$, we obtain

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*) Formula (14) appears in [2, p. 131, Equation (3.17)] with a misprint in the exponent of $(1+q)$.

(18)
$$\sum_{k=0}^{\infty}(-1)^k q^{k(k+2l-1)}\begin{bmatrix}\lambda-1\\ k\end{bmatrix}\frac{1-q^{\lambda+k+2l-2}}{(-q;q)_k(q^{\lambda+k};q)_{2l}}$$
$$=\frac{\Gamma_q(\lambda)}{(1-q)^{2l-1}}\left\{\frac{(1+q)^{\lambda}\Gamma_p(\lambda+l)}{\Gamma_q(2\lambda+2l-1)\Gamma_p(l)}-\frac{q^{\lambda+2l-2}(1+q)^{1-\lambda}\Gamma_p(l+1)}{\Gamma_p(\lambda+l)\Gamma_q(2l+1)}\right\}$$
$$(l\in N;\ p=q^2).$$

From the definitions (2) and (6), it follows readily that

(19)
$$\begin{bmatrix}n-1\\ k\end{bmatrix}=0\qquad (k=n,\ n+1,\ n+2,\ \cdots).$$

Thus, in the special case when $\lambda=n\in N$, each of the sums in (9) onwards would terminate at $k=n-1$, and we find from (18) and (12) that

(20)
$$\sum_{k=0}^{n-1}(-1)^k q^{k(k+2l+1)}\begin{bmatrix}n-1\\ k\end{bmatrix}\frac{1-q^{n+k+2l-2}}{(-q;q)_k(q^{n+k};q)_{2l}}$$
$$=\frac{(q;q)_{n-1}(q^2;q^2)_{n+l-1}}{(q;q)_{2n+2l-2}(q^2;q^2)_{l-1}}-\frac{(1-q)q^{n+2l-2}(q^2;q^2)_l}{(-q;q)_{n+l-1}(q;q)_{2l}(q^n;q)_l}\qquad (l,n\in N).$$

In particular, this last result (20) for $l=1$ yields

(21)
$$\sum_{k=0}^{n-1}(-1)^k q^{k(k+1)}\begin{bmatrix}n-1\\ k\end{bmatrix}\frac{1-q^n}{(1-q^{n+k+1})(-q;q)_k}$$
$$=\frac{(q;q)_n(q^2;q^2)_n}{(q;q)_{2n}}-\frac{q^n}{(-q;q)_n}\qquad (n\in N)$$

or, equivalently,

(22)
$$\sum_{k=0}^{n-1}(-1)^k q^{k(k+1)}\begin{bmatrix}n-1\\ k\end{bmatrix}\{(1-q^{n+k+1})(-q;q)_k\}^{-1}$$
$$=\frac{(q^2;q^2)_n}{(q^n;q)_{n+1}}-\frac{q^n}{(1-q^n)(-q;q)_n}\qquad (n\in N).$$

Formula (21) or (22) provides a q-extension of the Vowe-Seiffert sum (1); in fact, in the limit when $q\to 1$, (21) reduces immediately to (1). Formulas (16), (17), (18), and (20), on the other hand, provide q-extensions of our earlier results [4, p. 57, Equations (18) to (21)].

Finally, we record the following rather simple consequences of the general result (9) with $\lambda=n-1$ $(n\in N)$:

(23)
$$\sum_{k=0}^{n-1}(-1)^k q^{k^2}\begin{bmatrix}n-1\\ k\end{bmatrix}\{(-q;q)_k\}^{-1}=\{(-q;q)_{n-1}\}^{-1}\qquad (n\in N),$$

(24)
$$\sum_{k=0}^{n-1}(-1)^k q^{k(k+1)}\begin{bmatrix}n-1\\ k\end{bmatrix}\{(1-q^{n+k})(-q;q)_k\}^{-1}=\frac{(q^2;q^2)_n}{(q^n;q)_{n+1}}\qquad (n\in N),$$

and

(25)
$$\sum_{k=0}^{n-1}(-1)^k q^{k(k+2)}\begin{bmatrix}n-1\\ k\end{bmatrix}\{(1-q^{n+k})(1-q^{n+k+1})(-q;q)_k\}^{-1}$$
$$=\{(1-q)(1-q^n)(-q;q)_n\}^{-1}\qquad (n\in N).$$

The sum of the q-series in (21) or (22) would follow readily upon multiplying both sides of (25) by $(1-q)q^n$ and subtracting the resulting equation from (24). Formula (23), on the other hand, is an interesting companion of the basic (or q-) binomial theorem:

(26)
$$\sum_{k=0}^{n} q^{k(k-1)/2}\begin{bmatrix}n\\ k\end{bmatrix}a^k b^{n-k}=b^n(-a/b;q)_n\qquad (n\in N_0),$$

or, more generally,

$$\sum_{k=0}^{\infty} q^{k(k-2\lambda-1)/2} \begin{bmatrix} \lambda \\ k \end{bmatrix} z^k = \frac{(-zq^{-\lambda};q)_\infty}{(-z;q)_\infty} \qquad (|z|<1;\ \lambda \text{ arbitrary}). \tag{27}$$

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