# 45. *On Convolution Theorems*

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The object of the present paper is to prove convolution theorems for close-to-convex functions of order $\alpha$ and type $\beta$ and convex functions of order $\gamma$, and for functions satisfying $\mathrm{Re}\{f'(z)\}>\alpha$ and convex functions.

**1. Introduction.** Let $\mathcal{A}$ be the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U}=\{z:\ |z|<1\}$. We denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ the subclasses of $\mathcal{A}$ consisting of functions which are, respectively, starlike of order $\alpha$ $(0\leqq\alpha<1)$ in $\mathcal{U}$ and convex of order $\alpha$ $(0\leqq\alpha<1)$ in $\mathcal{U}$. In particular, we write $\mathcal{S}^*(0)\equiv\mathcal{S}^*$ and $\mathcal{K}(0)\equiv\mathcal{K}$.

A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be close-to-convex of order $\alpha$ and type $\beta$ if there exists a function $g(z)$ in the class $\mathcal{K}(\beta)$ such that

$$\mathrm{Re}\left\{\frac{f'(z)}{g'(z)}\right\}>\alpha \tag{1.2}$$

for some $\alpha$ $(0\leqq\alpha<1)$ and for all $z\in\mathcal{U}$. We denote by $\mathcal{K}_\alpha(\beta)$ the subclass of $\mathcal{A}$ consisting of functions which are close-to-convex of order $\alpha$ and type $\beta$ in $\mathcal{U}$. Also we write $\mathcal{K}_\alpha(0)\equiv\mathcal{K}_\alpha$.

Further, a function $f(z)$ in the class $\mathcal{A}$ is said to be a member of the class $\mathcal{R}(\alpha)$ if it satisfies

$$\mathrm{Re}\{f'(z)\}>\alpha \tag{1.3}$$

for some $\alpha$ $(0\leqq\alpha<1)$ and for all $z\in\mathcal{U}$.

For functions

$$f_j(z)=z+\sum_{n=2}^{\infty} a_{n,j} z^n \quad (j=1,2) \tag{1.4}$$

belonging to the class $\mathcal{A}$, we denote by $f_1 * f_2(z)$ the convolution (or Hadamard product) of functions $f_1(z)$ and $f_2(z)$, that is

$$f_1 * f_2(z)=z+\sum_{n=2}^{\infty} a_{n,1}a_{n,2} z^n. \tag{1.5}$$

**2. Convolution theorems.** In order to derive our convolution theorems, we have to recall here the following lemmas due to Owa [2].

**Lemma 1.** *Let* $\phi(z)\in\mathcal{K}$ *and* $g(z)\in\mathcal{S}^*$. *If* $F(z)\in\mathcal{A}$ *and* $\mathrm{Re}\{F(z)\}>\alpha$ $(0\leqq\alpha<1;\ z\in\mathcal{U})$, *then*

$$\mathrm{Re}\left\{\frac{\phi * G(z)}{\phi * g(z)}\right\}>\alpha \quad (z\in\mathcal{U}), \tag{2.1}$$

*where* $G(z)=F(z)g(z)$.

**Lemma 2.** *If $f(z) \in \mathcal{K}(\alpha)$ and $h(z) \in \mathcal{K}(\beta)$, then $h * f(z) \in \mathcal{K}(\gamma)$, where $\gamma = \max(\alpha, \beta)$.*

Applying the above lemmas, we prove

**Theorem 1.** *If $f(z) \in \mathcal{K}_\alpha(\beta)$ and $h(z) \in \mathcal{K}(\gamma)$, then $h * f(z) \in \mathcal{K}_\alpha(\delta)$, where $\delta = \max(\beta, \gamma)$.*

*Proof.* Note that, for $f(z) \in \mathcal{K}_\alpha(\beta)$, there exists a function $p(z) \in \mathcal{K}(\beta)$ such that

$$\operatorname{Re}\left\{\frac{f'(z)}{p'(z)}\right\} > \alpha \quad (z \in \mathcal{U}). \tag{2.2}$$

Letting $\phi(z) = h(z)$, $g(z) = zp'(z)$, $F(z) = f'(z)/p'(z)$, we have $\phi(z) \in \mathcal{K}(\gamma)$, $g(z) \in \mathcal{S}^*(\beta)$, and $\operatorname{Re}\{F(z)\} > \alpha$ $(z \in \mathcal{U})$. Therefore, using Lemma 1, we see that

$$\operatorname{Re}\left\{\frac{\phi * G(z)}{\phi * g(z)}\right\} = \operatorname{Re}\left\{\frac{h * zf'(z)}{h * zp'(z)}\right\} = \operatorname{Re}\left\{\frac{(h * f(z))'}{(h * p(z))'}\right\} > \alpha. \tag{2.3}$$

On the other hand, it follows from Lemma 2 that $h * p(z) \in \mathcal{K}(\delta)$, where $\delta = \max(\beta, \gamma)$. This implies that there exists a function $h * p(z) \in \mathcal{K}(\delta)$ such that

$$\operatorname{Re}\left\{\frac{(h * f(z))'}{(h * p(z))'}\right\} > \alpha \quad (z \in \mathcal{U}),$$

that is, that $h * f(z) \in \mathcal{K}_\alpha(\delta)$.

Next, we derive

**Theorem 2.** *If $f(z) \in \mathcal{R}(\alpha)$ and $h(z) \in \mathcal{K}$, then $h * f(z) \in \mathcal{K}$.*

*Proof.* Taking $\phi(z) = h(z) \in \mathcal{K}$, $g(z) = z \in \mathcal{K}$ and $F(z) = f'(z)$ in Lemma 1, we obtain that

$$\operatorname{Re}\left\{\frac{\phi * G(z)}{\phi * g(z)}\right\} = \operatorname{Re}\left\{\frac{h * zf'(z)}{z}\right\} = \operatorname{Re}\{(h * f(z))'\} > \alpha, \tag{2.4}$$

which shows that $h * f(z) \in \mathcal{R}(\alpha)$.

It is well-known by Strohhäcker [3] (also by MacGregor [1]) that if $f(z) \in \mathcal{K}$, then $f(z) \in \mathcal{S}^*(1/2)$. Therefore, in view of Theorem 1 and Theorem 2, we have the following conjectures.

**Conjecture 1.** *If $f(z) \in \mathcal{K}_\alpha(\beta)$ and $h(z) \in \mathcal{S}^*(1/2)$, then $h * f(z) \in \mathcal{K}_\alpha(\beta)$.*

**Conjecture 2.** *If $f(z) \in \mathcal{R}(\alpha)$ and $h(z) \in \mathcal{S}^*(1/2)$, then $h * f(z) \in \mathcal{R}(\alpha)$.*

## References

[1] T. H. MacGregor: A subordination for convex functions of order $\alpha$. J. London Math. Soc., (2)**9**, 530–536 (1975).

[2] S. Owa: A note on a convolution theorem. Chinese J. Math., **15**, 147–151 (1987).

[3] E. Strohhäcker: Beiträge zur theorie der schlichten Funktionen. Math. Z., **37**, 356–380 (1933).