5. *Some Aspects in the Theory of Representations of Discrete Groups. II*

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Here we concern mainly with equivalence relations among irreducible unitary representations (=IURs) of an infinite wreath product group, constructed in the first part [1] of these notes. We keep to the notations in [1].

1. Commutativity of two kinds of inducing processes. Let T be a group and S its subgroup. Consider wreath product groups $\mathfrak{S}_A(S)$ and $\mathfrak{S}_A(T)$. Then we have two kinds of inducing of representations: the usual one and the WP-inducing. We give a certain commutativity of these inducing processes. Start with a datum $R=\{A, \rho_S, \chi, a=(a_\alpha)_{\alpha\in A}\}$ for an elementary representation of $\rho(R)$ of $\mathfrak{S}_A(S)$. On the one hand, put $\tilde{\rho}_T=\mathrm{Ind}_S^T\rho_S$, and let $\tilde{a}_\alpha=\mathrm{Ind}_S^T a_\alpha\in V(\tilde{\rho}_T)$ be the induced vector of $a_\alpha\in V(\rho_S)$. Then $\tilde{a}=(\tilde{a}_\alpha)_{\alpha\in A}$ is a reference vector for $(\tilde{V}_\alpha)_{\alpha\in A}$ with $\tilde{V}_\alpha=V(\tilde{\rho}_T)$, and denote it as $\tilde{a}=\mathrm{Ind}_S^T a$. Thus we get a datum $\tilde{R}=\{A, \tilde{\rho}_T, \chi, \tilde{a}\}$ for $\mathfrak{S}_A(T)$ and correspondingly an elementary representation $\rho(\tilde{R})$ of $\mathfrak{S}_A(T)$. On the other hand, we have the induced representation $\mathrm{Ind}(\rho(R)\,;\,\mathfrak{S}_A(S)\uparrow\mathfrak{S}_A(T))$.

Theorem 1. *Let R be a datum for an elementary representation of $\mathfrak{S}_A(S)$. Then the two representations $\rho(\tilde{R})$ and $\mathrm{Ind}(\rho(R)\,;\,\mathfrak{S}_A(S)\uparrow\mathfrak{S}_A(T))$ of $\mathfrak{S}_A(T)$ are canonically equivalent to each other. A similar assertion holds for standard representation for $\mathfrak{S}_A(S)$ and $\mathfrak{S}_A(T)$.*

2. Equivalence relations among standard representations. Take two induced representations $\rho(Q_i)=\mathrm{Ind}(\pi(Q_i)\,;\,H(Q_i)\uparrow\mathfrak{S}_A(T))$, $i=1, 2$, of $\mathfrak{S}_A(T)$, called standard, and let the corresponding data be

$$Q_1=\{(A_\gamma, \rho_{T_{1\gamma}}^\gamma, \chi_{1\gamma})_{\gamma\in\Gamma}, (a_1(\gamma))_{\gamma\in\Gamma}, (b_{1\gamma})_{\gamma\in\Gamma}\},$$
$$Q_2=\{(B_\delta, \rho_{T_{2\delta}}^\delta, \chi_{2\delta})_{\delta\in\Delta}, (a_2(\delta))_{\delta\in\Delta}, (b_{2\delta})_{\delta\in\Delta}\},$$

where, in particular, $(A_\gamma)_{\gamma\in\Gamma}$ and $(B_\delta)_{\delta\in\Delta}$ are partitions of A, and $T_{1\gamma}$ and $T_{2\delta}$ are subgroups of T. For an element ζ of $\mathfrak{S}_A$, we call an *adjustment* of Q_2 by ζ the datum

$${}^\zeta Q_2=\{(\zeta(B_\delta), \rho_{T_{2\delta}}^\delta, \chi_\delta)_{\delta\in\Delta}, (a_2(\delta))_{\delta\in\Delta}, (b_{2\delta})_{\delta\in\Delta}\}.$$

Then $\rho(Q_2)$ is equivalent to $\rho({}^\zeta Q_2)$ in a trivial fashion.

Theorem 2. *Assume that two data Q_1 and Q_2 satisfy the condition (Q1), i.e., $|\Gamma_f|\leqq 1$, $|\Delta_f|\leqq 1$, and that both $\rho(Q_1)$ and $\rho(Q_2)$ are irreducible. Then they are mutually equivalent if and only if the following conditions hold.*

(EQU1) *Replacing Q_2 by its adjustment by an element in $\mathfrak{S}_A$ if necessary, we have a 1–1 correspondence κ of Γ onto Δ such that $A_\gamma=B_{\kappa(\gamma)}$ for $\gamma\in\Gamma$. Further $\chi_\gamma=\chi_{\kappa(\gamma)}$ for $\gamma\in\Gamma$, and $\mathrm{Ind}_{T_{1\gamma}}^T\rho_{T_{1\gamma}}^\gamma\cong\mathrm{Ind}_{T_{2\delta}}^T\rho_{T_{2\delta}}^\delta$ for $\gamma\in\Gamma_f$ and $\delta=\kappa(\gamma)$.*

(EQU2) *For* $\gamma\in\Gamma_\infty=\Gamma\setminus\Gamma_f$, *replace* $\delta=\kappa(\gamma)$ *by* γ, *and put* $T_{0\gamma}=T_{1\gamma}\cap T_{2\gamma}$. *Then, for every* $\gamma\in\Gamma_\infty$, *there exist an* IUR $\rho^\gamma_{T_{0\gamma}}$ *of* $T_{0\gamma}$ *and a reference vector* $a_0(\gamma)=(a_{0\alpha})_{\alpha\in A_\gamma}$, $a_{0\alpha}\in V(\rho^\gamma_{T_{0\gamma}})$, $\|a_{0\alpha}\|=1$, *such that for* $j=1, 2$, $\rho^\gamma_{T_{j\gamma}}\cong\mathrm{Ind}(\rho^\gamma_{T_{0\gamma}}; T_{0\gamma}\uparrow T_{j\gamma})$, *and* $a_j(\gamma)$ *is Moore-equivalent to the induced vector* $\mathrm{Ind}(a_0(\gamma); T_{0\gamma}\uparrow T_{j\gamma})$ *in the extended sense.*

(EQU3) *For* $\gamma\in\Gamma_\infty$, *put* $\chi_{0\gamma}=\chi_{1\gamma}$ $(=\chi_{2\gamma})$ *and*

$$Q_{j\gamma}=\{A_\gamma, \rho^\gamma_{T_{j\gamma}}, \chi_{j\gamma}, a_j(\gamma)\},\quad 0\leqq j\leqq 2,$$

and consider IURs $\Pi(Q_{j\gamma})$ *of* $H_{j\gamma}=\mathfrak{S}_{A_\gamma}(T_{j\gamma})$. *Then there exists a unit vector* $b_{0\gamma}\in V(\Pi(Q_{0\gamma}))$ *for every* $\gamma\in\Gamma_\infty$ *such that* $(b_{j\gamma})_{\gamma\in\Gamma_\infty}$, $j=1, 2$, *are respectively Moore-equivalent in the extended sense to* $(\tilde b_{j\gamma})_{\gamma\in\Gamma_\infty}$ *with* $\tilde b_{j\gamma}=\mathrm{Ind}(b_{0\gamma}; H_{0\gamma}\uparrow H_{j\gamma})$, *with respect to the representations* $\Pi(Q_{j\gamma})$ *and* $\mathrm{Ind}(\Pi(Q_{0\gamma}); H_{0\gamma}\uparrow H_{j\gamma})$.

Here note that, under the condition (EQU2), the IUR $\Pi(Q_{j\gamma})$ is equivalent to the induced one $\mathrm{Ind}(\Pi(Q_{0\gamma}); H_{0\gamma}\uparrow H_{j\gamma})$ for $j=1, 2$, by Theorem 1.

3. Fundamental lemmas for the proof. Put $G=\mathfrak{S}_A(T)$, $\pi_i=\pi(Q_i)$, $H_i=H(Q_i)$, then $\rho(Q_i)=\mathrm{Ind}^G_{H_i}\pi_i$. In the case where both π_i are finite-dimensional, Theorem 2 can be proved by means of the criterions in Theorem 1 in [1]. However, in the general case, we should appeal to the intertwining number equality (1) in [1], or more exactly we should study if there exists an $x\in G$ for which $d_x>0$, where d_x denotes the dimention of the space of $L\in\mathrm{Hom}(\pi_1, \pi_2{}^x; H_1\cap x^{-1}H_2x)$ satisfying the boundedness conditions (B_x) and (C_x). It needs heavy calculations but the lemmas used there are rather elementary. Here we give some fundamental ones.

Let F be a finite group, S a subgroup, and ρ an IUR of F. Put $V_1=V(\rho)$ and let V_2 be a unitary S-module. Take Hilbert spaces W_1, W_2, and consider $V_1\otimes W_1$ (resp. $V_2\otimes W_2$) as an F-module (resp. S-module) trivially. For an $L\in\mathrm{Hom}_S(V_1\otimes W_1, V_2\otimes W_2)$, we put for $u\in V_1\otimes W_1$,

$$J(u)=\sum_{f\in S\backslash F}\|L\rho(f)u\|^2=|S|^{-1}\sum_{f\in F}\|L\rho(f)u\|^2. \tag{1}$$

Then, detailed evaluations of this kind of sums are crucial for our purpose.

Denote by $\hat S$ the set of equivalence classes of IURs of S. For $\eta\in\hat S$, put $d(\eta)=\dim\eta$, $m(\rho,\eta)=[\rho|S:\eta]$, the multiplicity of η in $\rho|S$, and

$$\delta(\rho,\eta)=\frac{|F|\cdot d(\eta)}{|S|\cdot d(\rho)},\quad c(\rho,\eta)=\frac{\delta(\rho,\eta)}{m(\rho,\eta)}\qquad \text{if } m(\rho,\eta)>0.$$

Let $V_{i\eta}$ be the η-part of V_i as S-module and decompose it into irreducibles as $V_{i\eta}=\sum_l^\oplus V_{i\eta l}$, where $1\leqq l\leqq m(\rho,\eta)$ for $i=1$, and $1\leqq l\leqq m_2(\eta)\equiv$ the multiplicity of η in V_2, for $i=2$. Further let $J_{\eta;l'l}$ be a unitary S-isomorphism of $V_{1\eta l}$ onto $V_{2\eta l'}$. Then there exist $L^{\eta;l'l}\in\boldsymbol{B}(W_1, W_2)$ such that

$$L=\sum^\oplus_{\eta\in\hat S}L_\eta\qquad\text{with } L_\eta=\sum_{l'l}J_{\eta;l'l}\otimes L^{\eta;l'l}. \tag{2}$$

Lemma 3. (i) *Let* $u\in V_1\otimes W_1$ *and* $w\in W_1$, *then*

$$\sup_{\|u\|\leqq 1}J(u)=\sup_{\|w\|\leqq 1}\{\sum_\eta\delta(\rho,\eta)\cdot\sum_{l'l}\|L^{\eta;l'l}w\|^2\}. \tag{3}$$

(ii) *For* $\eta\in\hat S$ *such that* $m(\rho,\eta)>0$ *and the* η*-part* L_η *of* L,

$$\sup_{\|u\|\leqq 1}J(u)\geqq c(\rho,\eta)\cdot\|L_\eta\|^2. \tag{4}$$

Note that $\|L\|=\|L_\eta\|$ for some η.

Lemma 4. *For any $\eta\in\hat{S}$ such that $m(\rho,\eta)>0$, we have $\delta(\rho,\eta)\geqq c(\rho,\eta)\geqq 1$. Further $\delta(\rho,\eta)=1$ if and only if $\mathrm{Ind}_S^F\eta\cong\rho$; and $c(\rho,\eta)=1$ if and only if $\mathrm{Ind}_S^F\eta$ is equivalent to a multiple of ρ.*

4. Method of proof for Theorem 2. We can reduce the discussions on (B_x) and (C_x) to the case $x=e$.

(1°) We first apply the above lemmas to the following situation. From the data Q_1 and Q_2, we denote $T_{1\alpha}=T_{1\gamma}$, $\rho_{1\alpha}=\rho^{\gamma}_{T_{1\gamma}}$ for $\alpha\in A_\gamma$, $T_{2\alpha}=T_{2\delta}$, $\rho_{2\alpha}=\rho^{\delta}_{T_{2\delta}}$ for $\alpha\in B_\delta$, and $S_\alpha=T_{1\alpha}\cap T_{2\alpha}$, $V_{i\alpha}=V(\rho_{i\alpha})$. For a finite subset C of A, put

$$T_{iC}=\prod_{\alpha\in C}T_{i\alpha},\ \rho_{iC}=\otimes_{\alpha\in C}\rho_{i\alpha},\ V_{iC}=\otimes_{\alpha\in C}V_{i\alpha},\ S_C=\prod_{\alpha\in C}S_\alpha.$$

Then, in the sum (1), we take T_{1C} as F, ρ_{1C} as ρ, S_C as S, V_{iC} as V_i, and as W_i the tensor product of $V_{i\alpha}$, $\alpha\notin C$, so as to get $V(\pi_i)=V_i\otimes W_i$. Denote the corresponding sum $J(u)$ in (1) by $J_C(u)$. Now assume that L satisfies the condition (B_e). Then we get

$$J_C(u)\leqq M\|u\|^2 \qquad \text{for } u\in V(\pi_1).$$

Applying mainly the evaluation (4) in Lemma 3 and studying the growth of $J_C(u)$ as $|C|\to\infty$, we see the following. For every $\gamma\in\Gamma_\infty$, only the series $(\eta_\alpha)_{\alpha\in A_\gamma}$ with $\eta_\alpha\in\hat{S}_\alpha$ such that $c(\rho_{1\alpha},\eta_\alpha)=1$ for almost all $\alpha\in A_\gamma$, can intervene in the expression (2) of L, as one can expect it to avoid the divergence: $\prod_{\alpha\in C}c(\rho_{1\alpha},\eta_\alpha)\to\infty$. Also every reference vector $a(\gamma)$ in Q_1 should be equivalent to someone coming from the subspaces of $V_{1\alpha}$, $\alpha\in A_\gamma$, given as the sums of η_α-parts of $V_{1\alpha}$ with $c(\rho_{1\alpha},\eta_\alpha)=1$.

(2°) We also apply (C_e) for T_{2C}, ρ_{2C}, V_{2C} and S_C, and get the similar assertion for Q_2.

(3°) Next we proceed to take into account the condition (B_e) for $\prod'_{\gamma\in\Gamma}\mathfrak{S}_{A_\gamma}$ and the one (C_e) for $\prod'_{\delta\in\Delta}\mathfrak{S}_{B_\delta}$. This time we apply, together with (4), the more exact evaluation (3) of $J(u)$, and thus come to the condition $\delta(\rho_{1\alpha},\eta_\alpha)=1$ stronger than $c(\rho_{1\alpha},\eta_\alpha)=1$. Actually we should follow long calculations and discussions, to arrive at Theorem 2 finally.

Remark 5. We get in this way an explicit expression of an $L\in\mathrm{Hom}(\pi_1,\pi_2;H_1\cap H_2)$ satisfying (B_e) and (C_e), unique up to scalar multiples, and hence that of $T\in\mathrm{Hom}(\rho(Q_1),\rho(Q_2);\mathfrak{S}_A(T))$. This explicit form of intertwining operators will play important roles in our discussions on the unitary equivalences among the IURs of the infinite symmetric group $\mathfrak{S}_\infty$ which we construct using the results on IURs of wreath product groups.

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Note added in proof. It is regrettable that the first part [1] of the present notes should appear afterward.

Reference

[1] T. Hirai: Some aspects in the theory of representations of infinite discrete groups. I. (to appear).