4. *Estimates of Harmonic Measures Associated with Degenerate Laplacian on Strictly Pseudoconvex Domains*

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Let D be a smooth bounded strictly pseudoconvex domain in C^n, and let λ be a C^∞ strictly plurisubharmonic function on a neighborhood U of the closure $\bar{D}$ of D satisfying that $D=\{z\in U: \lambda(z)<0\}$, and $d\lambda\neq 0$ on the boundary ∂D of D. The purpose of the present paper is to announce our results on harmonic measures associated with the Laplace-Beltrami operator L of the complete Kähler metric $-\partial\bar{\partial}\log(-\lambda)$ on D. Detailed proofs will appear in a later paper.

Our main results are the following two theorems:

Theorem 1. *Let $z\in D$, and let $d\omega^z$ be the L-harmonic measure associated with L and D, evaluated at z, that is, $d\omega^z$ is a probability Borel measure on ∂D such that for any $f\in C(\partial D)$ the function $u(z)=\int f d\omega^z$ is the unique solution of $Lu=0$ in D which is continuous in $\bar{D}$ and satisfies that $u=f$ on ∂D. Then the measure $d\omega^z$ and the induced Euclidean measure $d\sigma$ on ∂D are mutually absolutely continuous. Furthermore, there exists a function k_z on ∂D such that $d\omega^z(\zeta)=k_z(\zeta)d\sigma(\zeta)$, $k_z\in L^\infty(d\sigma)$ and $k_z^{-1}\in L^\infty(d\sigma)$.*

From the point of view of several complex variables, the function $k_z(\zeta)$ ($z\in D$, $\zeta\in\partial D$) can be regarded as an analogue of the Poisson-Szegö kernel for the strictly pseudoconvex domain D. By this theorem and some results given later we obtain

Theorem 2. *Let u be an L-harmonic function on D. Let E be a subset of ∂D. If u is admissibly bounded at every point of $\zeta\in E$ in the sense of Stein* [11], *then u has an admissible limit at $d\sigma$-almost every point of E. (See* [11] *for the definition of admissible limits.)*

Since real and imaginary parts of holomorphlc functions are L-harmonic, Theorem 2 generalizes the local Fatou theorem for holomorphic functions (Stein [11, Theorem 12 (a)$\to$(b)]) to L-harmonic functions.

§1. On the proof of Theorem 1. In his paper [1], Ancona introduced the concept "Φ-chains", and stated in terms of Φ-chains a Harnack principle at infinity (see [1, Theorem 5′]). By modifying and localizing his theory, we can gain some boundary Harnack principles stated in terms of non-isotropic balls in ∂D and the normal vector field to ∂D. The proofs of the theorems involve the principles. Moreover, we need an estimate of L-harmonic measures. To describe it, let us recall some definitions and a basic fact: Let $D_0 \subset C^n$ be a domain contains ∂D such that for every $z \in D_0$ there exists the unique point $b(z)$ of ∂D with $|b(z)-z|=\delta(z)$, where $\delta(z)$ is the Euclidean distance between z and ∂D. For $z \in D \cap D_0$, let π_z be the orthogonal projection of the Euclidean space C^n onto the complex vector space spanned by the inward unit normal vector $\nu_{b(z)}$ to ∂D at $b(z)$, and let $\pi_z^\perp := I - \pi_z$, where I is the identity map. Then it is obvious that the non-isotropic ball of radius $r>0$, centered at $\zeta \in \partial D$ is equivalent to the following set:

$$Q(\zeta, r) := \{w \in \partial D : |\pi_\zeta(\zeta - w)| < r,\ |\pi_\zeta^\perp(\zeta - w)|^2 < r\}.$$

From now on we denote by g the metric $-\partial\bar{\partial}\log(-\lambda)$.

The following proposition is an analogue of the well known estimate of a uniformly elliptic harmonic measure:

Proposition 1. *For $\zeta \in \partial D$ and $r>0$, let $\zeta(r)=\zeta+r\nu_\zeta$. Then there exists a constant $c>0$ depending only on D and g such that*

$$\omega^{\zeta(r)}(Q(\zeta, r)) \geq c.$$

We are now in a position to prove Theorem 1: It is proved by the Harnack principles, Proposition 1, its localization, a theorem of Malliavin ([8, Theorem 2.1]) and a modification of Saks [9, Theorem 15.7].

§2. **On the proof of Theorem 2.** Let us recall the definition of admissible domains introduced by Stein [11]: For $\alpha>1$ and $\zeta \in \partial D$, let

$$A_\alpha(\zeta) := \{z \in D \cap D_0 : |\pi_\zeta(z-\zeta)| < \alpha\delta_\zeta(z),\ |z-\zeta|^2 < \alpha\delta_\zeta(z)\},$$

where $\delta_\zeta(z) = \min\{\delta(z), \operatorname{dist}(z, \mathrm{T}_\zeta(\partial D))\}$.

We will characterize the admissible domains by polydiscs: For $z \in D \cap D_0$ and for a small number $c>0$, let

$$P_c(z) := \{w \in D : |\pi_z(z-w)| < c\delta(z),\ |\pi_z^\perp(z-w)|^2 < c\delta(z)\},$$

and for $\zeta \in \partial D$, let

$$\Gamma(\zeta; c) := \cup\{P_c(\zeta + r\nu_\zeta) : r>0\} \cap D_0.$$

Our characterization is as follows:

Proposition 2. *We can take an open set $D_1 \subset C^n$ satisfying*

(i) $\partial D \subset D_1 \subset D_0$;

(ii) *For $\alpha>1$, there exist two positive constants $c(\alpha)$ and $C(\alpha)$ with*

$$\Gamma(\zeta; c(\alpha)) \cap D_1 \subset A_\alpha(\zeta) \cap D_1 \subset \Gamma(\zeta; C(\alpha)) \cap D_1, \qquad \zeta \in \partial D.$$

Theorem 2 is proved in the same spirit as the arguments given in [3] except using admissible domains introduced by Stein ([11]) instead of one defined in [3]. The proof is based on Theorem 1, Propositions 1, 2 and that aspect of the abstract potential theory which is related to the fine convergence.

§3. **Generalizations of theorems.** Let α be a positive constant. For $\Phi \in C^0(\bar{D}) \cap C^\infty(D)$ with $\Phi>0$ in the intersection of $\bar{D}$ and a neighborhood of ∂D, let

$$g(\alpha, \Phi) := -\alpha\partial\bar{\partial}\log(-\lambda\cdot\Phi).$$

Suppose that $g(\alpha, \Phi)$ is a complete Kähler metric of D and that its Laplace-Beltrami operator $L_{g(\alpha,\Phi)}$ is weakly coercive near ∂D in the sense of [1]. Theorem 1 is generalized as follows:

Theorem 3. *Suppose the Green's function G of $L_{g(\alpha,\Phi)}$ satisfies that for $z \in D$,*

$$C^{-1}\delta(w)^n \leq G(z, w) \leq C\delta(w)^n, \tag{G}$$

for all $w \in D$ near ∂D, where C is a positive constant independent of points w. Then $d\sigma$ and $L_{g(\alpha,\Phi)}$-harmonic measures are mutually absolutely continuous.

A typical example of metrics in Theorem 3 is the metric g (see [6], [7] and [8]). Another example is the Bergman metric of a certain strictly pseudoconvex domain (see [6], [7], [8] and [10]).

Theorem 2 is generalized to the same metric as in Theorem 3.

Remark. There are many results on absolute continuity of uniform elliptic harmonic measures (cf. [4], [5], [8]). Nevertheless, we can not use them, because L is degenerate at ∂D.

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References

[1] A. Ancona: Negatively curved manifolds, elliptic operators, and the Martin boundary. Ann. of Math., **125**, 495–539 (1987).
[2] M. T. Anderson and R. Schoen: Positive harmonic functions on complete manifolds of negative curvature. ibid., **121**, 429–539 (1985).
[3] H. Arai: Boundary behavior of functions on complete manifolds of negative curvature. Tohoku Math. J., **41**, 307–319 (1989).
[4] B. Dahlberg: On estimates of harmonic measure. Arch. Rat. Mech. Anal., **65**, 272–288 (1977).
[5] E. Fabes, D. Jerison, and C. Kenig: Necessary and sufficient conditions for absolute continuity of elliptic harmonic measure. Ann. of Math., **119**, 121–141 (1984).
[6] C. Fefferman: The Bergman kernel and biholomorphic mappings of pseudoconvex domains. Invent. Math., **26**, 1–65 (1974).
[7] P. F. Klembeck: Kähler metrics of negative curvature, the Bergman metric near the boundary, and the Kobayashi metric on smooth bounded pseudoconvex sets. Indiana Univ. Math. J., **27**, 275–282 (1978).
[8] P. Malliavin: Fonctions de Green d'un ouvert strictement pseudo-convexe et classe de Nevanlinna. C. R. Acad. Sci. Paris, **278**, 141–144 (1974).
[9] S. Saks: Theory of the Integral. 2nd ed., Hafner Publ. Co., New York (1937).
[10] T. Sasaki: On the Green function of a complete Riemannian or Kähler manifold with asymptotically negative constant curvature and applications. Advanced Studies in Pure Math., **3**, 387–421 (1984).
[11] E. M. Stein: Boundary Behavior of Holomorphic Functions of Several Complex Variables. Princeton Univ. Press (1972).