62. The Topological Invariant of Three-manifolds Based on the $U(1)$ Gauge Theory

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In this paper, we shall construct a topological invariants of three-manifolds using a series of representations of the Siegel modular group. We denote by $Sp(2g, Z)$ the Siegel modular group. By a series of projective unitary representations $\rho=\{\rho_g\}_{g\in N}$ of $Sp(Z)=\{Sp(2g, Z)\}_{g\in N}$, we shall mean that $\rho_g: Sp(2g, Z)\to PU(V_g)$ is a projective unitary representation on a hermitian vector space V_g for each $g\in N$.

We shall consider the following two conditions on ρ.

Condition 1. *There exists a hermitian vector space isomorphism φ_g: $V_g\otimes V_1\to V_{g+1}$ for each $g\in N$, satisfying*

$\bar{\varphi}_g(\rho_g(X), \rho_g(Y))=\rho_{g+1}(\iota_g(X, Y))$ *for $X\in Sp(2g, Z)$ and $Y\in Sp(2, Z)$, where $\bar{\varphi}_g: PU(V_g)\times PU(V_1)\to PU(V_{g+1})$ is the homomorphism induced by φ_g and $\iota_g: Sp(2g, Z)\times Sp(2, Z)\to Sp(2g+2, Z)$ is the natural inclusion map.*

Condition 2. *There exists an unit vector $v_0^{(1)}\in V_1$, and if we define $v_0^{(g)}\in V_g$ inductively by $v_0^{(g)}=\varphi_{g-1}(v_0^{(g-1)}\otimes v_0^{(1)})$, then $v_0^{(g)}$ is a simultaneous eigenvector of $\rho_g(\mathfrak{p}_g)$ $(\subset PU(V_g))$, where $\mathfrak{p}_g=\left\{\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in Sp(2g, Z)\,\middle|\, C=0\right\}$.*

If we are given a series of projective unitary representations $\rho=\{\rho_g\}_{g\in Z}$ of $Sp(Z)$ satisfying above Conditions 1 and 2, we can construct a $C/U(1)$ valued topological invariant $W(M)$ for a closed oriented three manifold M in the following manner.

We can represent M by a Heegaard splitting $M=H_g\cup_h(-H_g)$, where H_g is the handle body of genus g, and $h: \sum_g (=\partial H_g)\to\sum_g$ is an orientation preserving homeomorphism. We fix a basis $\{\lambda_i\}_{i=1}^{2g}$ of $H^1(\sum_g, Z)$ satisfying

$$(1)\qquad \begin{cases}\omega(\lambda_i, \lambda_j)=\omega(\lambda_{g+i}, \lambda_{g+j})=0 \\ \omega(\lambda_i, \lambda_{g+j})=\delta_{i,j},\end{cases}\qquad i, j=1, 2, \cdots, g,$$

where ω is the intersection form on $\sum_g$, and

$$(2)\qquad \operatorname{Im} i^*=\bigoplus_{i=1}^{g} Z\lambda_i,$$

where $i^*: H^1(\sum_g, Z)\to H^1(\sum_g, Z)$ is the homomorphism induced by the inclusion map $i: \sum_g\to H_g$.

Then we regard $\hat{h}=(h^{-1})^*: H^1(\sum_g, Z)\to H^1(\sum_g, Z)$ as an element of $Sp(2g, Z)$ with respect to the basis $\{\lambda_i\}_{i=1}^{2g}$.

Now we define

$$W(M)=c_0^{1-g}\langle\rho_g(\hat{h})v_0^{(g)}, v_0^{(g)}\rangle \qquad (\in C/U(1)),$$

and

$$c_0=\langle\rho_1(J_2)v_0^{(1)}, v_0^{(1)}\rangle \qquad (\in C/U(1)),$$

where $J_2=\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$.

The well-definedness of above definition is guaranteed by the following.

Theorem 1. *$W(M)$ does not depend on the particular Heegaard splitting of M. Therefore $W(M)$ is a topological invariant of M.*

Using the stabilization theorem due to Reidemeister and Singer ([2]), Theorem 1 easily follows from Conditions 1 and 2. And it is also easy to see that $W(M)$ does not depend on the particular basis $\{\lambda_i\}_{i=1}^{2g}$ satisfying (1) and (2).

An example of a series of projective unitary representations of $Sp(Z)$ satisfying Conditions 1 and 2 can be obtained by some geometric construction. More precisely we can construct the hermitian vector bundle $E^{(k)}$ over the Siegel's upper half plane $\mathfrak{H}_g$ for each $k\in 2N$, which has a natural projectively flat connection, and $Sp(2g, Z)$ acts on $E^{(k)}\to\mathfrak{H}_g$.

The monodromy representation of the above action $\rho_g^{(k)}$ can be given as follows. Let V_g be the hermitian vector space with the unitary basis $\{\Psi_l\}_{l\in I_g^{(k)}}$ parametrized by

$$I_g^{(k)}=\left\{l=\begin{pmatrix} l_1 \\ \vdots \\ l_g \end{pmatrix}\in Z^g \,\middle|\, 0\leq l_i<k,\ i=1,2,\cdots,g\right\}.$$

Note that $Sp(2g, Z)$ is generated by the elements of the forms $\begin{pmatrix} & I_g \\ -I_g & \end{pmatrix}$, $\begin{pmatrix} A & \\ & {}^tA^{-1} \end{pmatrix}$ and $\begin{pmatrix} I_g & B \\ & I_g \end{pmatrix}$, where $A\in GL(g, Z)$ and $B\in M_g(Z)$ with ${}^tB=B$. We define

$$\rho_g^{(k)}\begin{pmatrix} & I_g \\ -I_g & \end{pmatrix}\Psi_l=k^{-g/2}\sum_{l'\in I_g^{(k)}} e^{2\pi\sqrt{-1}k^{-1}{}^tl'\cdot l}\Psi_{l'},$$

$$\rho_g^{(k)}\begin{pmatrix} A & \\ & {}^tA^{-1} \end{pmatrix}\Psi_l=\sum_{l'\in I_g^{(k)}} a_{l'l}\Psi_{l'},$$

$$\text{where } a_{l'l}=\begin{cases} 1 & \text{if } {}^tAl'\equiv l \bmod kZ^g, \\ 0 & \text{if } {}^tAl'\not\equiv l \bmod kZ^g \end{cases}$$

and

$$\rho_g^{(k)}\begin{pmatrix} I_g & B \\ & I_g \end{pmatrix}\Psi_l=e^{\pi\sqrt{-1}k^{-1}{}^tlBl}\Psi_l.$$

Then the projective unitary representation $\rho_g^{(k)}: Sp(2g, Z)\to PU(V_g)$ is defined by these equations. And it is easy to see that $\rho^{(k)}=\{\rho_g^{(k)}\}_{g\in N}$ satisfies Conditions 1 and 2. Hence these give rise to a topological invariant $W_k(M)$ of M parametrized by $k\in 2N$.

Example.

$$W_k(S^1\times S^2)=1$$

$$W_k(S^3)=\frac{1}{k^{1/2}}$$

$$W_k(L_{p,1})=\frac{1}{k}\sum_{l=0}^{k-1} e^{-\pi\sqrt{-1}k^{-1}pl^2}$$

$$W_k(L_{2m+1,2})=\frac{1}{k^{3/2}}\sum_{l,l'=0}^{k-1} e^{-\pi\sqrt{-1}k^{-1}(ml^2-2l\cdot l'-2l'^2)}.$$

The detailed discussion and proofs will be given in the forthcoming paper.

References

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[3] E. Witten: Quantum field theory and the Jones polynomial. Comm. Math. Phys., **121**, 351–399 (1989).