77. *Regular Duo Elements of Abstract Affine Near-rings*

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1. Introduction. In his paper [2], Steinfeld characterizes the regular duo elements of a ring in terms of quasi-ideals.

The purpose of this note is to extend the above result to a class of abstract affine near-rings. For the basic terminology and notation we refer to [1].

2. Preliminaries. Let N be a near-ring, which always means right one throughout this note.

If A, B and C are three non-empty subsets of N, then AB (ABC) denotes the set of all finite sums of the form $\sum a_k b_k$ with $a_k \in A$, $b_k \in B$ ($\sum a_k b_k c_k$ with $a_k \in A$, $b_k \in B$, $c_k \in C$), and $A*B$ denotes the set of all finite sums of the form $\sum (a_k(a'_k+b_k)-a_k a'_k)$ with a_k, $a'_k \in A$, $b_k \in B$.

A right N-subgroup (left N-subgroup) of N is a subgroup H of $(N, +)$ such that $HN \subseteq H$ ($NH \subseteq H$). For an element n of N, $(n)_r$ ($(n)_l$) denotes the principal right (left) N-subgroup of N generated by n. A quasi-ideal of N is a subgroup Q of $(N, +)$ such that $QN \cap NQ \cap N*Q \subseteq Q$. Right N-subgroups and left N-subgroups are quasi-ideals. The intersection of a family of quasi-ideals is again a quasi-ideal.

An element n of N is called regular if $n=nxn$ for some element x of N, and the element n is said to be a duo element of N, if $(n)_r=(n)_l$.

3. Main results. A near-ring N is called an abstract affine near-ring if N is abelian and $N_0=N_d$, where N_0 and N_d are the zero-symmetric part and the set of all distributive elements of N, respectively.

Lemma 1. *Let a be an element of an abstract affine near-ring N. Then the product $(a)_l(a)_r$ is a two-sided N-subgroup of N.*

Proof. It is clear that the product $(a)_l(a)_r$ is a subgroup of $(N, +)$. By the definitions we get

$$((a)_l(a)_r)N \subseteq (a)_l((a)_r N) \subseteq (a)_l(a)_r,$$

whence $(a)_l(a)_r$ is a right N-subgroup of N.

On the other hand, the left N-subgroup $(a)_l$ contains the constant part N_c of N. So the product $(a)_l(a)_r$ also contains N_c. Hence it follows that

$$\begin{aligned} N((a)_l(a)_r) &= (N_0+N_c)((a)_l(a)_r) = N_0((a)_l(a)_r)+N_c \\ &= (N_0(a)_l)(a)_r+N_c \subseteq (a)_l(a)_r+N_c \subseteq (a)_l(a)_r, \end{aligned}$$

whence $(a)_l(a)_r$ is a left N-subgroup of N.

Lemma 2 ([3, Theorem]). *The following assertions concerning an* *element a of an abstract affine near-ring N are equivalent:*

(1) *a is regular.*

(2) $(a)_r(a)_l=(a)_r\cap(a)_l$.

(3) $(a)_r^2=(a)_r$, $(a)_l^2=(a)_l$ *and the product* $(a)_r(a)_l$ *is a quasi-ideal of* N.

Now we are ready to state the main results of this note.

Theorem 1. *The following conditions on an element a of an abstract affine near-ring N are equivalent:*

(α) *a is a regular duo element of N.*

(β) $(a)_l(a)_r=(a)_l\cap(a)_r$

(γ) $(a)_r^2=(a)_l$ *and* $(a)_l^2=(a)_r$.

Proof. $(\alpha)\Rightarrow(\gamma)$: As a is a duo element, $(a)_r=(a)_l$. This and the condition (3) in Lemma 2 imply (γ).

$(\gamma)\Rightarrow(\beta)$: From the assumption ($\gamma$), it follows

$$(a)_l\cap(a)_r=(a)_l\cap(a)_l^2=(a)_l^2=(a)_l(a)_l=(a)_l(a)_r^2\subseteq(a)_l(a)_r.$$

On the other hand,

$$(a)_l(a)_r=\begin{cases}(a)_l(a)_l^2\subseteq(a)_l,\\(a)_r^2(a)_r\subseteq(a)_r,\end{cases}$$

whence $(a)_l(a)_r\subseteq(a)_l\cap(a)_r$.

$(\beta)\Rightarrow(\alpha)$: From $a\in(a)_l\cap(a)_r=(a)_l(a)_r$ and Lemma 1, it follows that the principal left N-subgroup $(a)_l$ is contained in the two-sided N-subgroup $(a)_l(a)_r$, whence

$$(a)_l\subseteq(a)_l(a)_r=(a)_l\cap(a)_r\subseteq(a)_r.$$

Similarly $(a)_r\subseteq(a)_l(a)_r=(a)_l\cap(a)_r\subseteq(a)_l$. Thus $(a)_l=(a)_r$ and a is a duo element. This and the condition (β) imply the property (2) in Lemma 2. So a is a regular element.

A near-ring N is called a duo near-ring if every one-sided (right or left) N-subgroup of N is a two-sided N-subgroup of N. From Theorem 1 and [4, Propositions 1 and 2], one gets immediately

Theorem 2. *Let N be an abstract affine near-ring. Then the following conditions are equivalent:*

(A) *N is a regular duo near-ring.*

(B) *N is a regular duo ring.*

(C) *Every element a of N has one of the properties* (α), (β) *and* (γ) *in Theorem* 1.

4. Remark. The following example shows that Theorem 1 can not be extended to arbitrary near-rings: Let $N=\{0,1,2,3,4,5\}$ be the near-ring due to [1, Near-rings of low order (G-13)] defined by the tables on next page.

Then $(2)_r=(2)_l=\{0,2,4\}$, and the assertions (β) and (γ) hold for the element 2. But the element 2 is not a regular duo element of N, since $2x2=0$ for all elements x of N.

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

·	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	4	0	0	4	0
2	0	2	0	0	2	0
3	0	0	0	0	0	0
4	0	4	0	0	4	0
5	0	2	0	0	2	0

References

[1] G. Pilz: Near-rings. 2nd ed., North-Holland, Amsterdam (1983).

[2] O. Steinfeld: Notes on regular duo elements, rings and semigroups. Studia. Sci. Math. Hung., **8**, 161–164 (1973).

[3] I. Yakabe: Regular elements of abstract affine near-rings. Proc. Japan Acad., **65A**, 307–310 (1989).

[4] ——: Regular duo near-rings. ibid., **66A**, 115–118 (1990).