15. *Multiple Sine Functions and Selberg Zeta Functions*

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We describe basic properties of multiple sine functions, and as an application, we report the calculation of the "gamma factors" of Selberg-Gangolli-Wakayama zeta functions of rank one locally symmetric spaces:

Theorem. *The gamma factors are products of multiple gamma functions of Barnes, and they are expressed as determinants of the Laplace operators of the compact dual symmetric spaces. Moreover Plancherel measures are sums of logarithmic derivatives of multiple gamma functions.*

This result has been known for the case of a compact Riemann surface by Vignéras [11] and Cartier-Voros [3]. The typical example of our result is the case of an even dimensional real hyperbolic space $M=\Gamma\backslash G/K$ for $G=\mathrm{SO}(1,2n)$ and $K=\mathrm{SO}(2n)$. Let $Z_M(s)$ be the Selberg zeta function. Then the gamma factor $\Gamma_M(s)$ is given by

$$\begin{aligned}\Gamma_M(s)&\cong(\Gamma_{2n}(s)\Gamma_{2n}(s+1))^{-\mathrm{vol}(M)(-1)^n}\\&\cong\det\left(\left(\Delta_{S^{2n}}+\left(n-\frac{1}{2}\right)^2\right)^{1/2}+s-n+\frac{1}{2}\right)^{\mathrm{vol}(M)(-1)^n}\end{aligned}$$

and the completed zeta function $\hat{Z}_M(s)=Z_M(s)\Gamma_M(s)$ is invariant under the transformation $s\to 2n-1-s$:

$$\hat{Z}_M(s)\cong\det\left(\Delta_M+s\left(s-n+\frac{1}{2}\right)\right),$$

where $\Delta_{S^{2n}}$ and Δ_M are Laplace operators with non-negative eigenvalues. Here, the above isomorphisms indicate that we have omitted factors $\exp(P(s))$ for polynomials $P(s)$ (whose determination would not be difficult for experts). Our result would suggest to investigate the behaviour of $Z_M(s)$ as $\dim(M)\to\infty$. (We may consider zeta functions of filtered Riemannian spaces.) The fact that "gamma factors" can be written by multiple gamma functions seems to have a philosophical meaning (cf. [6] [7]). Some details will appear elsewhere.

§1. **Multiple sine functions.** We recall basic facts concerning multiple sine functions studied in a previous paper [8]. For a positive integer r and a complex number z, we put

$$P_r(z)=(1-z)\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^r}{r}\right).$$

We define the multiple gamma function $G_r(z)$ and the multiple sine function $F_r(z)$ of order $r\geqq 2$ as follows:

$$G_r(z)=\exp\left(\frac{-(-z)^{r-1}}{2(r-1)}\right)\prod_{n=1}^{\infty}P_r\left(-\frac{z}{n}\right)^{-n^{r-1}}$$

and

$$\begin{aligned}F_r(z)&=G_r(z)^{(-1)^r}G_r(-z)^{-1}\\&=\exp\left(\frac{z^{r-1}}{r-1}\right)\prod_{n=1}^{\infty}\left(P_r\left(\frac{z}{n}\right)P_r\left(-\frac{z}{n}\right)^{(-1)^{r-1}}\right)^{n^{r-1}}.\end{aligned}$$

This $G_r(z)$ is a simplified version of the multiple gamma function $\Gamma_r(z)$ of Barnes [1] and we use simple relations between them later. For $r=1$, it is convenient to put

$$G_1(z)=(2\pi)^{-1/2}\Gamma(z) \quad \text{and} \quad F_1(z)=G_1(z)^{-1}G_1(1-z)^{-1}=2\sin(\pi z)$$

using the ordinary gamma function $\Gamma(z)$.

The multiple sine function $F_r(z)$ $(r\geqq 2)$ is characterised by each of the following properties (as a meromorphic function):

(1) $$F_r'(z)=\pi z^{r-1}\cot(\pi z)F_r(z) \quad \text{and} \quad F_r(0)=1,$$

(2) $$F_r(z)=\exp\left(\int_0^z \pi t^{r-1}\cot(\pi t)dt\right),$$

(3) $$F_r''(z)=(1-z^{1-r})F_r'(z)^2F_r(z)^{-1}+(r-1)z^{-1}F_r'(z)-\pi^2z^{r-1}F_r(z)$$
with $F_r(0)=1$ and $F_r'(0)=1$ $(r=2)$ or 0 $(r\geqq 3)$,

(4) $$F_r(z)=\exp\left(-\frac{(r-1)!}{(2\pi i)^{r-1}}\sum_{k=0}^{r-1}\frac{(2\pi i)^k}{k!}z^k\operatorname{Li}_{r-k}(e^{-2\pi i z})+\frac{\pi i}{r}z^r+\frac{(r-1)!}{(2\pi i)^{r-1}}\zeta(r)\right)$$

in $\operatorname{Im}(z)<0$ using the polylogarithm function $\operatorname{Li}_k(x)=\sum_{n=1}^{\infty}x^n n^{-k}$. These can be easily seen from the definition as in [8]: (1) is the most basic and (2)–(4) follow from it. We use the expression (2) to calculate gamma factors of Selberg zeta functions as described in § 2 below. It may be interesting to notice that the second order algebraic differential equation (3) is similar to the Painlevé's differential equation of type III. We can express the special values $\zeta(2m+1)$ for $m=1,2,\cdots$ using $F_{2k+1}(1/2)$ for $k\leqq m$ as an application of (4) (and its analogue for $\operatorname{Im}(z)\geqq 0$). Following are typical examples:

$$\zeta(3)=\frac{8\pi^2}{7}\log\left(\frac{2^{1/4}}{F_3(1/2)}\right)$$

and

$$\zeta(5)=\frac{32\pi^4}{93}\log\left(\frac{F_5(1/2)2^{11/112}}{F_3(1/2)^{9/14}}\right).$$

The above expression of $\zeta(3)$ is equivalent to the formula of Euler [4, § 21] written in 1772:

$$1+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\cdots=\frac{\pi^2}{4}\log 2+2\int_0^{\pi/2}\varphi\log(\sin\varphi)d\varphi.$$

In fact, the left hand side is $(7/8)\zeta(3)$ and by integration by parts the right hand side is equal to

$$\frac{\pi^2}{4}\log 2-\pi^3\int_0^{1/2}t^2\cot(\pi t)dt=\frac{\pi^2}{4}\log 2-\pi^2\log F_3\left(\frac{1}{2}\right).$$

Values $\zeta(2m+1)$ are inductively obtained from the following formula:

$$\zeta(2m+1)=(-1)^m \frac{(4\pi)^{2m}}{(2m)!\,(2^{2m+1}-1)} \log (F_{2m+1}(1/2)2^{-1/4m})$$
$$-\sum_{k=1}^{m-1} \frac{(-1)^k(2\pi)^{2k}(4^{m-k}-1)}{(2k)!\,(2^{2m+1}-1)} \zeta(2m-2k+1).$$

The study of Shintani [10] would suggest to investigate the algebraicity of $F_r(z)$ for rational numbers z. We know that $F_1(1/2)=2$ and $F_2(1/2)=\sqrt{2}$, and $F_{2m}(1/2)$ are expressed by $F_{2k-1}(1/2)$ for $k \leqq m$ as indicated by the following example: $F_4(1/2)=F_3(1/2)^{9/14}2^{-1/28}$. The algebraicity of $F_r(z)$ for rational numbers z would follow from $F_r(m)=0$ for $m=1,2,\cdots$ when we have a suitable addition formula or a multiplication formula for $F_r(z)$. It is not difficult to express $F_r(mz)$ via products of $F_r(z+k/m)$ for $k=0,\cdots,m-1$, but, this "multiplication formula" seems to be not so useful to the algebraicity. In the case of $r=2$, this duplication formula is

$$F_2(2z)=F_2(z)^2F_2(z+1/2)^2F_1(z+1/2)^{-1},$$

which is analogous to the usual duplication formula of the sine function: $F_1(2z)=F_1(z)F_1(z+1/2)$. Notice that $F_1(z+1/2)=2\cos(\pi z)$, so we may consider $F_r(z+1/2)$ as the multiple cosine function of order r.

§ 2. **Gamma factors of Selberg zeta functions.** We describe the main points of the proof of Theorem. Let $M=\Gamma\backslash G/K$ be a compact locally symmetric space of rank one. It is sufficient to treat the case $\operatorname{rank} G=\operatorname{rank} K$, since otherwise ($G=SO(1,2n+1)$) the "gamma factor" is trivial by the nonexistence of the discrete series. We denote by $Z_M(s;\sigma,\tau)$ the Selberg zeta function of M constructed by Wakayama [12] (generalising Selberg [9] and Gangolli [5]) associated with $(\sigma,\tau)\in \operatorname{Rep}(\Gamma)\times\operatorname{Rep}(K)$, where Rep denotes the set of finite dimensional unitary representations of a group. Wakayama [12, Th. 7.2] shows that $Z_M(s;\sigma,\tau)$ is meromorphic on $\boldsymbol{C}$, and that it has the functional equation:

$$Z_M(2\rho_0-s;\sigma,\tau)=Z_M(s;\sigma,\tau)\exp\left(\int_0^{s-\rho_0} \Delta_{\sigma,\tau}(t)dt\right)$$

where $\Delta_{\sigma,\tau}$ is essentially given by the Plancherel density with K-types. Our problem is to express the exponential factor as $\Gamma_M(s;\sigma,\tau)/\Gamma_M(2\rho_0-s;\sigma,\tau)$ by the "gamma factor" $\Gamma_M(s;\sigma,\tau)$ so that the completed zeta function $\hat{Z}_M(s;\sigma,\tau)=Z_M(s;\sigma,\tau)\Gamma_M(s;\sigma,\tau)$ is invariant under $s\to 2\rho_0-s$. Explicit formulas for $\Delta_{\sigma,\tau}$ (Wakayama [12; (6.9), (6.5), (6.2) and § 1 (II)-(V)]) reduce this problem to the calculation of the following two integrals:

$$I_r(z)=\int_0^z \pi t^{r-1}\cot(\pi t)dt \quad\text{and}\quad J_r(z)=\int_0^z \pi t^{r-1}\tan(\pi t)dt$$

for even integers $r\geqq 2$. We have seen that $I_r(z)=\log(G_r(z)/G_r(-z))$ and the integral $J_r(z)$ is inductively calculated by the following formula:

$$J_r(z)+\frac{(r-1)(r-2)}{8}J_{r-2}(z)+\cdots+\frac{r-1}{2^{r-2}}J_2(z)$$
$$=\frac{1}{2}\int_0^z \pi\left(\left(t+\frac{1}{2}\right)^{r-1}+\left(t-\frac{1}{2}\right)^{r-1}\right)\tan(\pi t)dt$$

$$=-\frac{1}{2}\log\left(\frac{G_r(z+1/2)G_r(z-1/2)}{G_r(-z+1/2)G_r(-z-1/2)}\right).$$

Thus, the gamma factors of Selberg zeta functions $Z_M(s;\sigma,\tau)$ are expressed as products of G_r. To express them via Γ_r, it is sufficient to notice that

$$G_r(z)\cong\prod_{k=1}^{r}\Gamma_k(z)^{c(r,k)}$$

for integers $c(r,k)$ such as $c(r,r)=(r-1)!$. The expression using the Laplace operator of the dual symmetric space is essentially a proportionality principle. In general the gamma factor is expressed as

$$\Gamma_M(s;\sigma,\tau)\cong\det\left((\Delta^{\tau}_{M'}+\rho_0^2)^{1/2}+s-\rho_0\right)^{\mathrm{vol}(M)\dim(\sigma)(-1)^{\dim(M)/2}},$$

where $M'=G'/K$ is the compact dual symmetric space, $\Delta^{\tau}_{M'}$ is the associated Laplace operator, and $\mathrm{vol}(M)$ is a suitably normalised volume. For simplicity we notice the case $M=\Gamma\backslash\mathrm{SO}(1,2n)/\mathrm{SO}(2n)$ for trivial σ and τ, where we see that

$$\Gamma_M(s)\cong\det\left(\left(\Delta_{S^{2n}}+\left(n-\frac{1}{2}\right)^2\right)^{1/2}+s-n+\frac{1}{2}\right)^{\mathrm{vol}(M)(-1)^n}$$
$$\cong(\Gamma_{2n}(s)\Gamma_{2n}(s+1))^{-\mathrm{vol}(M)(-1)^n}.$$

First we obtain the second equality from the result of Cartan [2] giving multiplicities of eigenvalues of Laplace operators of spheres. Then the first equality is proved by looking at the multiplicities of zeros and poles of both sides using our previous expression for $\Gamma_M(s)$ via G_r; both sides are meromorphic functions of order $2n$. The general case is exactly similar.

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