30. On Regular Subalgebras of a Symmetrizable Kac-Moody Algebra

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Let $\mathfrak{g}(A)$ be a Kac-Moody algebra with A a symmetrizable generalized Cartan matrix (= GCM) over the complex number field $\boldsymbol{C}$. In this paper, we study its certain subalgebras called *regular subalgebras*. These subalgebras are defined as a natural infinite dimensional analogue of *regular semi-simple subalgebras* of a finite dimensional complex semi-simple Lie algebra in the sense of Dynkin. The latter plays an important role in the classification of semi-simple subalgebras (cf. [1]).

§1. Definition of regular subalgebras. Let A be an $n\times n$ symmetrizable GCM, and $\mathfrak{h}$ be a Cartan subalgebra of the Kac-Moody algebra $\mathfrak{g}(A)$. Then we have the root space decomposition of $\mathfrak{g}(A)$:

$$\mathfrak{g}(A)=\mathfrak{h}\oplus\sum\nolimits_{\alpha\in\Delta}^{\oplus}\mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha=\{x\in\mathfrak{g}(A)\,;\,[h,x]=\langle\alpha,h\rangle x,\ \text{for all } h\in\mathfrak{h}\}$ for $\alpha\in\mathfrak{h}^*$ (the algebraic dual of $\mathfrak{h}$), and $\Delta\subset\mathfrak{h}^*$ is the root system of $\mathfrak{g}(A)$ (see [3] for details). To define a *regular subalgebra* of $\mathfrak{g}(A)$, we introduce the notion of *fundamental* subset of Δ.

Definition 1.1. A subset $\bar{\Pi}=\{\beta_1,\cdots,\beta_m,\beta_{m+1},\cdots,\beta_{m+k}\}$ of the root system Δ of $\mathfrak{g}(A)$ is called *fundamental* if it satisfies the following:

(1) $\bar{\Pi}=\{\beta_r\}_{r=1}^{m+k}$ is a linearly independent subset of $\mathfrak{h}^*$;

(2) $\beta_s-\beta_t\notin\Delta\cup\{0\}$ $(1\le s\ne t\le m+k)$;

(3) β_i is a real root $(1\le i\le m)$ and β_j is a positive imaginary root $(m+1\le j\le m+k)$.

Now, let $(\cdot|\cdot)$ be a fixed standard invariant form on $\mathfrak{g}(A)$ such that $(\alpha_i|\alpha_j)\in\boldsymbol{Z}$ $(1\le i,j\le n)$, where $\{\alpha_i\}_{i=1}^n\subset\Delta$ is the set of all simple roots of $\mathfrak{g}(A)$ (cf. [3, Chap. 2]). For each imaginary root β_j $(m+1\le j\le m+k)$, we define $\beta_j^\vee:=\nu^{-1}(\beta_j)\in\mathfrak{h}$, where $\nu:\mathfrak{h}\to\mathfrak{h}^*$ is a linear isomorphism determined by $\langle\nu(h),h'\rangle=(h|h')$ $(h,h'\in\mathfrak{h})$. For real root β_i $(1\le i\le m)$, $\beta_i^\vee\in\mathfrak{h}$ has been defined as a dual real root of β_i, and we know $\beta_i^\vee=2/(\beta_i|\beta_i)\cdot\nu^{-1}(\beta_i)$ (cf. [3, Chap. 5]).

Proposition 1.1. *Let* $\bar{\Pi}=\{\beta_r\}_{r=1}^{m+k}$ *be a fundamental subset of* Δ, *and put* $\bar{A}:=(\bar{a}_{ij})_{i,j=1}^{m+k}$, *where* $\bar{a}_{ij}=\langle\beta_j,\beta_i^\vee\rangle$. *Then,* $\bar{A}$ *is a symmetrizable GGCM (= generalized GCM). Moreover,* $\bar{a}_{ii}=2$ *if and only if* β_i *is a real root* $(1\le i\le m+k)$.

Here, $\bar{A}$ is a GGCM means that $\bar{A}$ satisfies the following:

(C1) either $\bar{a}_{ii}=2$ or $\bar{a}_{ii}$ is a non-positive integer;

(C2) $\bar{a}_{ij}$ is a non-positive integer if $i\ne j$;

(C3) $\bar{a}_{ij}=0$ implies $\bar{a}_{ji}=0$.

Note that when $\bar{a}_{ii}=2$ for every i, $\bar{A}$ is a GCM.

Let $\mathfrak{g}(\bar{A})$ be the Lie algebra associated to the above GGCM $\bar{A}$ (see [3, Chaps. 1 and 11]). We call it a generalized Kac-Moody algebra (= GKM algebra). Note that when $\bar{A}$ is a GCM, $\mathfrak{g}(\bar{A})$ is a Kac-Moody algebra by definition.

Proposition 1.2. *There exists a vector subspace $\mathfrak{h}_0$ of $\mathfrak{h}$, such that the triple $(\mathfrak{h}_0, \{\beta_r|\mathfrak{h}_0\}_{r=1}^{m+k}, \{\beta_r^\vee\}_{r=1}^{m+k})$ is a realization of the GGCM $\bar{A}$. That is, it satisfies the following conditions:*

(R1) *both the sets $\{\beta_r|\mathfrak{h}_0\}_{r=1}^{m+k}\subset\mathfrak{h}_0^*$ and $\{\beta_r^\vee\}_{r=1}^{m+k}\subset\mathfrak{h}_0$ are linearly independent;*

(R2) $\langle\beta_j, \beta_i^\vee\rangle=\bar{a}_{ij}$ $(1\leq i, j\leq m+k)$;

(R3) $\dim_{\boldsymbol{C}}\mathfrak{h}_0=2(m+k)-\operatorname{rank}\bar{A}$.

We fix non-zero vectors $E_r\in\mathfrak{g}_{\beta_r}$ and $F_r\in\mathfrak{g}_{-\beta_r}$ such that $[E_r, F_r]=\beta_r^\vee$ $(1\leq r\leq m+k)$. Note that such vectors always exist since $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]=\boldsymbol{C}\nu^{-1}(\alpha)$ for all $\alpha\in\Delta$. Let $\bar{\mathfrak{g}}$ be a subalgebra of $\mathfrak{g}(A)$ generated by E_r, F_r $(1\leq r\leq m+k)$, and a vector subspace $\mathfrak{h}_0$ of $\mathfrak{h}$ which satisfies (R1)–(R3). We call this kind of subalgebra a *regular subalgebra* of $\mathfrak{g}(A)$.

Theorem 1.1. *Any regular subalgebra of $\mathfrak{g}(A)$ is canonically isomorphic to a GKM algebra. Let $\bar{\mathfrak{g}}$ be as above. Then, a canonical isomorphism Φ of a GKM algebra $\mathfrak{g}(\bar{A})$ onto $\bar{\mathfrak{g}}$ is given as:*

$$\Phi(\bar{e}_r)=E_r,\quad \Phi(\bar{f}_r)=F_r\quad (1\leq r\leq m+k),\quad \text{and}\quad \Phi(\bar{\mathfrak{h}})=\mathfrak{h}_0.$$

Here $(\bar{\mathfrak{h}}, \{\bar{\alpha}_r\}_{r=1}^{m+k}, \{\bar{\alpha}_r^\vee\}_{r=1}^{m+k})$ is a realization of the GGCM $\bar{A}$, and $\bar{e}_r, \bar{f}_r$ $(1\leq r\leq m+k)$ are the Chevalley generators of the GKM algebra $\mathfrak{g}(\bar{A})$.

Remark 1.1. In the above theorem, we adopt the definition in [3, Chap. 11] of GKM algebras, which is a little different from that of Borcherds in [1]. As seen above, regular subalgebras are always isomorphic to GKM algebras, but not necessarily isomorphic to Kac-Moody algebras in general.

Remark 1.2. The above definition of a fundamental subset $\bar{\Pi}$ of Δ and the construction of a subalgebra $\bar{\mathfrak{g}}$ of $\mathfrak{g}(A)$ corresponding to $\bar{\Pi}$ are generalizations of those by Morita [5]. There, he considered only the case all β_r are real roots (i.e., $k=0$ in the above definition) and constructed a subalgebra $\hat{\mathfrak{g}}$, which coincides with the derived algebra $[\bar{\mathfrak{g}}, \bar{\mathfrak{g}}]$ of the above $\bar{\mathfrak{g}}$.

Remark 1.3. The subalgebra $\bar{\mathfrak{g}}$ depends on the choice of the vector subspace $\mathfrak{h}_0$ of $\mathfrak{h}$ satisfying (R1)–(R3). However, its derived algebra $[\bar{\mathfrak{g}}, \bar{\mathfrak{g}}]$ does not depend on the choice of $\mathfrak{h}_0$.

Proposition 1.3. *We have the following two decompositions of $\bar{\mathfrak{g}}$:*

(I) $\bar{\mathfrak{g}}=\sum_{\alpha\in Q_+\setminus\{0\}}^{\oplus}(\bar{\mathfrak{g}}\cap\mathfrak{g}_\alpha)\oplus(\bar{\mathfrak{g}}\cap\mathfrak{h})\oplus\sum_{\alpha\in Q_+\setminus\{0\}}^{\oplus}(\bar{\mathfrak{g}}\cap\mathfrak{g}_{-\alpha})$,

(II) $\bar{\mathfrak{g}}=\sum_{\beta\in\bar{Q}_+\setminus\{0\}}^{\oplus}(\bar{\mathfrak{g}}\cap\mathfrak{g}_\beta)\oplus\mathfrak{h}_0\oplus\sum_{\beta\in\bar{Q}_+\setminus\{0\}}^{\oplus}(\bar{\mathfrak{g}}\cap\mathfrak{g}_{-\beta})$,

with $Q_+:=\sum_{i=1}^n \boldsymbol{Z}_{\geq 0}\alpha_i$ and $\bar{Q}_+:=\sum_{r=1}^{m+k}\boldsymbol{Z}_{\geq 0}\beta_r$. Moreover for every $\beta\in\bar{Q}:=\sum_{r=1}^{m+k}\boldsymbol{Z}\beta_r$, we have $\bar{\mathfrak{g}}\cap\mathfrak{g}_\beta=\bar{\mathfrak{g}}_\beta$, where $\bar{\mathfrak{g}}_\beta:=\{x\in\bar{\mathfrak{g}}; [h, x]=\langle\beta, h\rangle x$, for all $h\in\mathfrak{h}_0\}$. Here we identify $\beta_r\in\mathfrak{h}^*$ with $\beta_r|\mathfrak{h}_0\in\mathfrak{h}_0^*$ (since $\{\beta_r|\mathfrak{h}_0\}_{r=1}^{m+k}\subset\mathfrak{h}_0^*$ is linearly independent).

From the above proposition, we can regard the root system $\bar{\Delta}$ of $\mathfrak{g}(\bar{A})$ $\cong\bar{\mathfrak{g}}$ as a subset of the root system Δ of $\mathfrak{g}(A)$, by the identification of $\beta_r|\bar{\mathfrak{h}}_0$ with β_r $(1\leq r\leq m+k)$, because $\bar{\Delta}$ is a subset of $\sum_{r=1}^{m+k}\boldsymbol{Z}(\beta_r|\bar{\mathfrak{h}}_0)$. Under this identification, we have the following:

$$\bar{\Delta}=\{\beta\in\Delta\,;\,\bar{\mathfrak{g}}\cap\mathfrak{g}_\beta\neq\{0\}\}.$$

Definition 1.2 (cf. [5]). $\bar{\Delta}$ is called a root subsystem of Δ.

§ 2. **The inheritance of a standard invariant form.** In this section, we assume that a fundamental subset $\bar{\Pi}$ consists of real roots (i.e., $k=0$ in Definition 1.1). So, the matrix $\bar{A}=(\langle\beta_j,\beta_i^\vee\rangle)_{i,j=1}^m$ is a GCM and the subalgebra $\bar{\mathfrak{g}}\cong\mathfrak{g}(\bar{A})$ is a Kac-Moody algebra. In this situation, we can take a "good" vector subspace $\bar{\mathfrak{h}}_0$ of $\mathfrak{h}$ as a vector subspace $\mathfrak{h}_0$ in Theorem 1.1 as shown below. Let $\bar{\Pi}=\{\beta_1,\cdots,\beta_m\}$ be a fundamental subset consisting of real roots and $\bar{A}=(\langle\beta_j,\beta_i^\vee\rangle)_{i,j=1}^m$. We put $l:=\operatorname{rank}A$ and $t:=\operatorname{rank}\bar{A}$, then clearly, $t\leq l$ and $t\leq m$.

Proposition 2.1. *There exists a basis* $\{h_i\}_{i=1}^{m+N}\cup\{v_j\}_{j=1}^{m-t}$ *of* $\mathfrak{h}$, *such that the presentation matrix R of the standard invariant form* $(\cdot|\cdot)$ *on* $\mathfrak{g}(A)$ *with respect to this basis is of the form*

$$R=\begin{bmatrix} J_1 & O & O & O \\ O & O_{m-t} & O & I_{m-t} \\ O & O & J_2 & O \\ O & I_{m-t} & O & O_{m-t} \end{bmatrix},$$

where I_{m-t} is the identity matrix of degree $m-t$, O_{m-t} is the zero matrix of degree $m-t$, $J_1=\operatorname{diag}(\pm1,\pm1,\cdots,\pm1)$: *$t\times t$-matrix, and* $J_2=\operatorname{diag}(\pm1,\pm1,\cdots,\pm1)$: *$N\times N$-matrix with* $N:=(2n-l)-(2m-t)(\geq 0)$.

Now let $\bar{\mathfrak{h}}_0:=\sum_{i=1}^m\boldsymbol{C}h_i+\sum_{j=1}^{m-t}\boldsymbol{C}v_j$. Then, we have the following.

Proposition 2.2. *The triple* $(\bar{\mathfrak{h}}_0,\{\beta_i|\bar{\mathfrak{h}}_0\}_{i=1}^m,\{\beta_i^\vee\}_{i=1}^m)$ *is a realization of the GCM* $\bar{A}$.

Let $\bar{\mathfrak{g}}$ be a subalgebra of $\mathfrak{g}(A)$ generated by E_r,F_r $(1\leq r\leq m)$, and the above $\bar{\mathfrak{h}}_0$. Then, we see from Theorem 1.1 that $\bar{\mathfrak{g}}$ is canonically isomorphic to a Kac-Moody algebra $\mathfrak{g}(\bar{A})$. Moreover, we can prove the following theorem thanks to the construction of $\bar{\mathfrak{h}}_0$ in such a detailed way as above.

Theorem 2.1. *Let* $\bar{\mathfrak{g}}\subset\mathfrak{g}(A)$ *be a regular subalgebra constructed from the above* $\bar{\mathfrak{h}}_0$. *Put* $\bar{B}:=((\beta_i|\beta_j))_{i,j=1}^m$ *and* $\bar{D}:=\operatorname{diag}(2/(\beta_1|\beta_1),\cdots,2/(\beta_m|\beta_m))$, *where* $(\cdot|\cdot)$ *is the fixed standard invariant form on* $\mathfrak{g}(A)$. *Then, the restriction of* $(\cdot|\cdot)$ *to* $\bar{\mathfrak{g}}\subset\mathfrak{g}(A)$ *coincides with a standard invariant form on* $\bar{\mathfrak{g}}$, *which is canonically identified with* $\mathfrak{g}(\bar{A})$.

This standard invariant form on $\bar{\mathfrak{g}}\cong\mathfrak{g}(\bar{A})$ is determined by the following:

(F1) $(\beta_i^\vee|h):=\langle\beta_i,h\rangle\cdot2/(\beta_i|\beta_i)$ $(h\in\bar{\mathfrak{h}}_0, 1\leq i\leq m)$,

(F2) $(h'|h''):=0$ $(h',h''\in\sum_{j=1}^{m-t}\boldsymbol{C}v_j)$,

(F3) $([x,y]|z)=(x|[y,z])$ $(x,y,z\in\bar{\mathfrak{g}})$.

Here this form, viewed from $\mathfrak{g}(\bar{A})$, corresponds to the decomposition $\bar{A}=\bar{D}\bar{B}$ and to a complementary subspace $\sum_{j=1}^{m-t}\boldsymbol{C}v_j$ to $\sum_{i=1}^m\boldsymbol{C}\beta_i^\vee$ in $\bar{\mathfrak{h}}_0$.

We denote by Δ^{re} (resp. Δ^{im}) the set of all real (resp. imaginary) roots of $\mathfrak{g}(A)$. Correspondingly, we denote by $\bar{\Delta}^{re}$ (resp. $\bar{\Delta}^{im}$) the set of all real (resp. imaginary) roots for the root system $\bar{\Delta}$ of $\mathfrak{g}(\bar{A})$. Then, we have the following as a direct consequence of Theorem 2.1.

Theorem 2.2. *For the root system $\bar{\Delta}$ of $\mathfrak{g}(\bar{A})$ $(\cong \bar{\mathfrak{g}})$, regarded as a root subsystem of Δ, we have*

$$\bar{\Delta}^{re}=\bar{\Delta}\cap\Delta^{re}, \qquad \bar{\Delta}^{im}=\bar{\Delta}\cap\Delta^{im}.$$

§ 3. Type of the GGCM $\bar{A}=(\langle\beta_j, \beta_i^\vee\rangle)_{i,j=1}^{m+k}$.

3.1. Some generalities. As an application of Theorem 2.1, we obtain the following theorem.

Theorem 3.1. *Let $A=(a_{ij})_{i,j=1}^{n}$ be a GCM of affine type, and $\bar{\Pi}=\{\beta_r\}_{r=1}^{m+k}$ be a fundamental subset of Δ. Put $\bar{A}:=(\langle\beta_j, \beta_i^\vee\rangle)_{i,j=1}^{m+k}$. Then, we have either of the following two cases:*

Case (a). *$\bar{\Pi}$ is contained in Δ^{re}, and $\bar{A}$ is a direct sum of GCM's of finite type or of affine type. Moreover, the number of direct summands of affine type is at most one.*

Case (b). *$\bar{\Pi}$ contains exactly one imaginary root, and $\bar{A}$ is a direct sum of the zero matrix O_1 of degree 1 (with multiplicity one) and GCM's of finite type.*

Remark 3.1. Note that the derived algebra of the Lie algebra $\mathfrak{g}(O_1)$ associated to the 1×1 GGCM O_1 is a Heisenberg Lie algebra ([3, Chap. 2]).

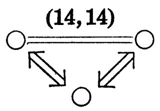
Contrary to this affine case, we have the following example for hyperbolic case.

Example 3.1. Let A be a 3×3-matrix given below. Then A is a GCM of hyperbolic type with the Dynkin diagram below.

$$A=\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \qquad \circ \Leftarrow\!\Rightarrow \circ \text{---} \circ .$$

Put $\beta_1:=(r_3r_2)(\alpha_1)$, $\beta_2:=r_1(\beta_1)$, and $\beta_3:=r_2(\beta_2)$, where r_i is a fundamental reflection defined by a simple root $\alpha_i\in\Delta$ $(1\leq i\leq 3)$. Put $\bar{\Pi}:=\{\beta_1, \beta_2, \beta_3\}\subset\Delta^{re}$. Then, $\bar{\Pi}$ is a fundamental subset. The corresponding GCM $\bar{A}$ and its Dynkin diagram are as follows.

$$\bar{A}=\begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & -14 \\ -2 & -14 & 2 \end{bmatrix},$$



Obviously, $\bar{A}$ is neither of finite type, of affine type, nor of hyperbolic type. (See [4] for a similar example.)

3.2. Case of affine type GCM. In this subsection, we assume that the GCM $A=(a_{ij})_{i,j=0}^{l}$ is of non-twisted affine type (cf. [3, Chaps. 4 and 6]). So, there exists $\delta=(a_i)_{i=0}^{l}$ such that $A\delta=0$ and $a_i\in\boldsymbol{Z}_{\geq 1}$ for all i $(0\leq i\leq l)$. Such a δ is unique under the condition that a_i $(0\leq i\leq l)$ are relatively prime. We take such a δ, and also denote $\sum_{i=0}^{l}a_i\alpha_i$ by δ. Then, we know the following facts:

$$\Delta^{im}=\{k\delta\,;\,k\in \boldsymbol{Z}\setminus\{0\}\},\qquad \Delta^{re}=\{\gamma+k\delta\,;\,\gamma\in\mathring{\Delta},\,k\in\boldsymbol{Z}\},$$

where $\mathring{\Delta}$ is the root system of the finite type Kac-Moody algebra $\mathfrak{g}(\mathring{A})\subset\mathfrak{g}(A)$ associated to the principal submatrix $\mathring{A}:=(a_{ij})_{i,j=1}^{l}$ of A. Note that the removed vertex 0 of the Dynkin diagram of A is so chosen that $a_0=1$ and the type of $\mathring{A}$ is X_l when the type of A is $X_l^{(1)}$ $(X=A, B, \cdots, G)$. Here we have the following theorem.

Theorem 3.2. *Let $A=(a_{ij})_{i,j=0}^{l}$ be a GCM of non-twisted affine type. Then, the Dynkin diagram of the GGCM $\bar{A}$ corresponding to a fundamental subset $\bar{\Pi}$ of Δ is of type either O_1, $X_{t_1}+X_{t_2}+\cdots+X_{t_r}$, $X_{t_1}^{(1)}+X_{t_2}+\cdots+X_{t_r}$, $X_{t_1}+X_{t_2}^{(1)}+\cdots+X_{t_r}$, $\cdots$, or $X_{t_1}+X_{t_2}+\cdots+X_{t_r}^{(1)}$, where $X_{t_1}+X_{t_2}+\cdots+X_{t_r}$ is the type of Dynkin diagram of the GCM corresponding to a fundamental subset of the root system $\mathring{\Delta}$ of $\mathfrak{g}(\mathring{A})$.*

Conversely, for each of the above types, there exists a fundamental subset of Δ whose Dynkin diagram is of that type.

Here X_{t_i} is the type of a finite type GCM of rank t_i, and O_1 denotes also the type of 1×1 GGCM O_1.

Note that when A is of non-twisted affine type, Case (b) in Theorem 3.1 does not happen except for the trivial case that $\bar{\Pi}$ consists of only one imaginary root. Owing to the above theorem, we can determine all the types of regular subalgebras (= the types of the GGCM's corresponding to fundamental subsets of Δ) of the non-twisted affine Lie algebra $\mathfrak{g}(A)$. This is because those of the finite dimensional simple Lie algebra $\mathfrak{g}(\mathring{A})$ are completely determined (see [2, Chap. II, §5]).

Remark 3.2. Also in the case of twisted affine type GCM, but not of type $A_{2l}^{(2)}$ $(l\geq 1)$, the sufficiency part (the second part) of Theorem 3.2 is true. Here note that for the GCM $A=(a_{ij})_{i,j=0}^{l}$ of type $A_{2l-1}^{(2)}$ $(l\geq 3)$, $D_{l+1}^{(2)}$ $(l\geq 2)$, $E_6^{(2)}$ or $D_4^{(3)}$, the type of $\mathring{A}=(a_{ij})_{i,j=1}^{l}$ is C_l, B_l, F_4, or G_2, respectively.

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