35. Fermat Motives and the Artin-Tate Formula. II

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4. The Artin-Tate-Milne formula for Fermat motives. Throughout the section, $k=\boldsymbol{F}_q$ of characteristic p and X *denotes the Fermat variety of dimension* $n=2r$ *and of degree* m *over* k.

If M is a Γ-module, then M^{Γ} and M_{Γ} denote respectively the kernel and cokernel of $\varphi-1: M\to M$, where φ is the canonical generator of $\Gamma=\mathrm{Gal}(\bar{k}/k)$.

If M is a finite group, $|M|$ denotes its order. For a prime number l, $|\ |_l$ denotes the l-adic absolute value normalized so that $|l|_l^{-1}=l$.

For a $(\boldsymbol{Z}/m)^{\times}$-orbit $A\subset\mathfrak{A}$, we define $w^{r-1}(A)=\sum_{\boldsymbol{a}\in A}\max(r-\|\boldsymbol{a}\|, 0)$.

Theorem 4.1 (The Artin-Tate-Milne formula I) (cf. [10], Ch. III. 2). *Let M_A be the Fermat submotive of X, corresponding to a $(\boldsymbol{Z}/m)^{\times}$-orbit $A\subset\mathfrak{A}$. If M_A is not supersingular, then*:

(a) $H^n(M_A, \boldsymbol{Z}_l(r))=0$ *for each prime l with $(l, mp)=1$*;

(b) $H^{n+1}(M_A, \boldsymbol{Z}_l(r))$ *is finite and* $|H^{n+1}(M_A, \boldsymbol{Z}_l(r))|=|P_A(1/q^r)|_l^{-1}$ *for each prime l with $(l, mp)=1$*;

(c) $H^n(M_A, \boldsymbol{Z}_p(r))=0$;

(d) $H^{n+1}(M_A, \boldsymbol{Z}_p(r))$ *is finite and* $|H^{n+1}(M_A, \boldsymbol{Z}_l(r))|=|P_A(1/q^r)|_p^{-1}\cdot q^{w^{r-1}(A)}$.

Combining the assertions of 4.1 with Iwasawa's theorem [12] and Remark 4.3, we obtain the following assertion:

Corollary 4.2. *Assume that m is a prime ≥ 3 and that $k=\boldsymbol{F}_q$ contains all the m-th roots of unity. Let M_A be a Fermat submotive of X, corresponding to a $(\boldsymbol{Z}/m)^{\times}$-orbit $A\subset\mathfrak{A}$. If M_A is not supersingular but of Hodge-Witt type, then*

$$Nr_{\boldsymbol{Q}(\zeta_m)/\boldsymbol{Q}}\left(1-\frac{j(\boldsymbol{a})}{q^r}\right)=\prod_{\boldsymbol{a}\in A}\left(1-\frac{j(\boldsymbol{a})}{q^r}\right)=\pm Bm^3/q^{w^{r-1}(A)},$$

where B is a positive integer which is a square, possibly multiplied by a divisor of $2m$.

Remark 4.3. Let X be a smooth projective variety of dimension $n=2r$ over k. Then the Bockstein operator $\beta: H^n(X, \boldsymbol{Q}_l/\boldsymbol{Z}_l(r))\to H^{n+1}(X, \boldsymbol{Z}_l(r))$ induces a bijection $\beta: H^n(X, \boldsymbol{Q}_l/\boldsymbol{Z}_l(r))_{\mathrm{cotors}}\xrightarrow{\sim} H^{n+1}(X, \boldsymbol{Z}_l(r))_{\mathrm{tors}}$. We define a bi-additive form on $H^{n+1}(X, \boldsymbol{Z}_l(r))_{\mathrm{tors}}$ with values in $\boldsymbol{Q}_l/\boldsymbol{Z}_l$ by $\langle x, y\rangle=x\cap\beta^{-1}(y)$, where $\cap$ denotes the cup-product pairing

$$H^{n+1}(X, \boldsymbol{Z}_l(r))\times H^n(X, \boldsymbol{Q}_l/\boldsymbol{Z}_l(r))\longrightarrow \boldsymbol{Q}_l/\boldsymbol{Z}_l.$$

Then $\langle\ ,\ \rangle$ is non-degenerate and skew-symmetric. Hence $|H^{n+1}(X, \boldsymbol{Z}_l(r))_{\mathrm{tors}}|$ is a square, or twice a square if $l=2$.

If M_A is a Fermat submotive of the Fermat variety X of dimension $n=2r$ and of degree m and l is a prime with $(l, m)=1$, the pairing

$$\langle\,,\,\rangle : H^{n+1}(X, Z_l(r))_{\mathrm{tors}} \times H^{n+1}(X, Z_l(r))_{\mathrm{tors}} \longrightarrow Q_l/Z_l$$

induces a non-degenerate skew-symmetric pairing

$$\langle\,,\,\rangle ; H^{n+1}(M_A, Z_l(r))_{\mathrm{tors}} \times H^{n+1}(M_A, Z_l(r))_{\mathrm{tors}} \longrightarrow Q_l/Z_l.$$

Remark 4.4. Shioda has proved that if $n=2$, $p\equiv 1 \pmod{m}$ and M_A is not supersingular, then

$$Nr_{Q(\zeta_m)/Q}\left(1-\frac{j(a)}{q}\right)=\prod_{a\in A}\left(1-\frac{j(a)}{q}\right)=\pm Bm^3/q^{w^0(A)},$$

where B is a positive integer which is a square up to $2mp$ (cf. [4], Th. 7.1). In this case, M_A is ordinary. He also suggested the assertion of 4.2.

4.5. To analyze the case when M_A is a supersingular Fermat submotive of X, we recall Milne's argument ([14], Prop. 6.5 and Prop. 6.6).

Let $\theta_l \in H^1(k, Z_l)=Z_l$ be a canonical generator and let $\varepsilon_{l,A}$ denote the homomorphism $H^n(M_A, Z_l(r)) \to H^{n+1}(M_A, Z_l(r))$ defined by the cup-product with θ_l. Then the diagram

$$\begin{array}{ccc} H^n(M_A, Z_l(r)) & \xrightarrow{\varepsilon_{l,A}} & H^{n+1}(M_A, Z_l(r)) \\ \downarrow \wr & & \uparrow \\ H^n(M_{A,\bar{k}}, Z_l(r))^{\Gamma} & \longrightarrow & H^n(M_{A,\bar{k}}, Z_l(r))_{\Gamma} \end{array}$$

is commutative. Here the vertical arrows are defined by the Hochschild-Serre spectral sequence

$$E_2^{r,j}=H^r(\Gamma, H^j(M_{A,\bar{k}}, Z_l(r)) \Longrightarrow H^{r+j}(M_A, Z_l(r))$$

and the horizontal arrow below is the composite of the obvious maps $H^n(M_{A,\bar{k}}, Z_l(r))^{\Gamma} \to H^n(M_{A,\bar{k}}, Z_l(r))$ and $H^n(M_{A,\bar{k}}, Z_l(r)) \to H^n(M_{A,\bar{k}}, Z_l(r))_{\Gamma}$.

Theorem 4.6 (The Artin-Tate-Milne formula II). *Assume that F_q contains all the m-th root of unity. Let M_A be a supersingular Fermat submotive of X.*

(1) *Let l be a prime with $(l, mp)=1$. Then $H^{n+1}(M_A, Z_l(r))$ is torsion-free and all the maps in the diagram*

$$\begin{array}{ccc} H^n(M_A, Z_l(r)) & \xrightarrow{\varepsilon_{l,A}} & H^{n+1}(M_A, Z_l(r)) \\ \downarrow & & \uparrow \\ H^n(M_{A,\bar{k}}, Z_l(r))^{\Gamma} & \longrightarrow & H^n(M_{A,\bar{k}}, Z_l(r))_{\Gamma} \end{array}$$

are bijective.

(2) $\varepsilon_{p,A} : H^n(M_A, Z_p(r)) \longrightarrow H^{n+1}(M_A, Z_p(r))$ *is injective and*

$$|\det \varepsilon_{p,A}|_p^{-1}\cdot|H^{2r+1}(M_A, Z_p(r))_{\mathrm{tors}}|=q^{w^{r-1}(A)}.$$

Corollary 4.7. *Assume that M_A is ordinary and supersingular. Then $H^{n+1}(M_A, Z_p(r))$ is torsion-free and all the maps in the diagram*

$$\begin{array}{ccc} H^n(M_A, Z_p(r)) & \xrightarrow{\varepsilon_{p,A}} & H^{n+1}(M_A, Z_p(r)) \\ \downarrow & & \uparrow \\ H^n(M_{A,\bar{k}}, Z_p(r))^{\Gamma} & \longrightarrow & H^n(M_{A,\bar{k}}, Z_p(r))_{\Gamma} \end{array}$$

are bijective.

4.8. Let $N^r(X)$ denote the image of the composite $CH^r(X) \to CH^r(X_{\bar{k}}) \to N^r(X_{\bar{k}})$.

Now we assume that

(1) The Tate conjecture holds true for X;

(2) The cycle map $CH^r(X)\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_l\to H^n(X_{\bar{k}}, \boldsymbol{Z}_l(r))^{\Gamma}$ is surjective for all primes l with $(l, m)=1$.

Proposition 4.9.1. *Under the above assumptions,* $\det N^r(X)$ *divides a power of mp. Moreover, if X is ordinary, i.e.* $p\equiv 1$ (mod. m), $\det N^r(X)$ *divides a power of m.*

Corollary 4.9.2. *Under the above assumptions,* $\det N^r(X_{\bar{k}})$ *divides a power of mp. Moreover, if X is ordinary,* $\det N^r(X_{\bar{k}})$ *divides a power of m.*

5. Examples. In this section, we assume that $k=\boldsymbol{F}_q$ contains all the m-th roots of unity.

5.1. Let X be the Fermat surface of degree m over k. Then we have $CH^1(X_{\bar{k}})=Pic(X_{\bar{k}})=N^1(X_{\bar{k}})=NS(X_{\bar{k}})$, the Néron-Severi group of $X_{\bar{k}}$, and $N^1(X)=NS(X)=NS(X_{\bar{k}})^{\Gamma}$. It is known that the Tate conjecture holds true for X (Tate [6], Shioda-Katsura [5]). Therefore the canonical maps $NS(X)\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_l\to H^2(X, \boldsymbol{Z}_l(1))$ and $Br(X)_{l\text{-tors}}\to H^3(X, \boldsymbol{Z}_l(1))_{\text{tors}}$ are bijective for each prime l, and $Br(X)$ is finite (Tate [18], Milne [13]). Here $Br(X)=H^2(X, \boldsymbol{G}_{m,X})$ denotes the Brauer group of X.

Theorem 5.2 (The Artin-Tate formula) ([10], Ch. IV. 2). *Let X be the Fermat surface of degree m and* M_A *the Fermat submotive of X, corresponding to a* $(\boldsymbol{Z}/m)^{\times}$*-orbit* $A\subset\mathfrak{A}$.

(I) *Suppose that* M_A *is not supersingular. Then:*

(a) $|Br(M_A)_{l\text{-tors}}|=\left|\prod_{\boldsymbol{a}\in A}\left(1-\frac{j(\boldsymbol{a})}{q}\right)\right|_l^{-1}$ *for each prime l with* $(l, mp)=1$.

(b) $|Br(M_A)_{p\text{-tors}}|/q^{w^0(A)}=\left|\prod_{\boldsymbol{a}\in A}\left(1-\frac{j(\boldsymbol{a})}{q}\right)\right|_p^{-1}$.

(II) *Suppose that* M_A *is supersingular. Then:*

(a) $Br(M_A)_{l\text{-tors}}=0$ *and* $|\det NS(M_A)\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_l|_l=1$ *for each prime l with* $(l, mp)=1$.

(b) $|Br(M_A)_{p\text{-tors}}|\,|\det NS(M_A)\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_p|_p^{-1}=q^{w^0(A)}$.

(Note that $w^0(A)=\sharp\{\boldsymbol{a}\in A;\ \|\boldsymbol{a}\|=0\}$.)

Remark 5.3. The assertions for the l-part with $(l, mp)=1$ in the above theorem are due to Shioda [4], Prop. 6.1.

Corollary 5.4. *If X is ordinary, then* $\det NS(X)$ *and* $\det NS(X_{\bar{k}})$ *divide a power of m.*

Remark 5.5. If $m=7$ and $p\equiv 2$ or 4 (mod. 7), X is of Hodge-Witt type (but not ordinary). It is similarly seen that $NS(X)$ and $NS(X_{\bar{k}})$ divide a power of $m=7$.

Remark 5.6. Shioda has proved that $\det NS(X)$ divides a power of mp for the Fermat surface X of degree m over $\boldsymbol{F}_q$ ([4], Cor. 6.3). He has also remarked that $\det NS(X_{\bar{k}})$ divides a power of m if X is ordinary (loc. cit. Remark 6.4 and Addendum).

5.7. Let X be the Fermat variety of dimension $n=2r\geq 4$ and of degree m over k. It is known that the Tate conjecture holds true for X in the following cases:

(1) X is ordinary, i.e. $p\equiv 1$ (mod. m) and m is not divisible by any prime less than $n+2$;

(2) X is ordinary, i.e. $p\equiv 1$ (mod. m) and m is a prime or 4;

(3) X is supersingular, i.e. $p^{\nu}\equiv -1$ (mod. m) for some ν

(Shioda [15], [16], Shioda-Katsura [5], Aoki [11]).

Proposition 5.8. *Let X be the Fermat variety of dimension $n=2r\geq 4$ and of degree m. Assume that* (1) *or* (2) *is satisfied. Then* $\det N^r(X)$ *and* $\det N^r(X_{\bar{k}})$ *divide a power of m.*

Corollary 5.9. *Let X_C be the Fermat variety of dimension $n=2r\geq 4$ and of degree m over* **C**. *Assume that*: (1) *m is not divisible by any prime less than $n+2$, or* (2) *m is a prime or* 4. *Then* $\det N^r(X_C)$ *divides a power of m.*

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