59. *Cubic Interpolatory Spline Matching the Areas**)

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(Communicated by Shokichi IYANAGA, M. J. A., Sept. 12, 1991)

1. Introduction. Schoenberg in his landmark paper [5] has given an ingenious solution to the problem of smoothing of histograms (cf. [1] also). Given a histogram on a uniform partition of an interval, it is shown in [5] that there exists a unique quadratic spline defined over the partition such that area underlying a polynomial-piece of the spline function matches the area of the corresponding histogram-cell. DeBoor [2] has considered this area matching condition (AMC) for even degree splines. Sharma and Tzimbalario [6] have studied quadratic splines which are such that in each subinterval of the partition, integral mean of the spline, with respect to a non-negative measure $d\mu$ matches with the same mean of a given function. By suitable choices of the measure function μ, it is possible to reduce this integral matching condition (IMC) into different interesting interpolatory conditions. In particular, when $\mu(x)=x$, the above interpolatory condition reduces to AMC. Similar integral matching condition has been studied for the case of cubic splines, by Dikshit [3]. However, the restrictions used in [3] on IMC do not allow the special choice of the measure function when IMC reduces to the AMC. The object of this paper is to investigate the existence, uniqueness and convergence properties of a cubic interpolatory spline satisfying area matching condition.

2. The cubic interpolatory spline. Let $P=\{0=x_0<x_1<\cdots<x_n=1\}$ be a uniform partition of interval [0, 1] so that $x_i-x_{i-1}=p$, $i=1, 2, \cdots, n$. Let f be a locally integrable 1-periodic function defined over [0, 1]. Let $S(4, P)$ denote the space of piecewise cubic polynomial spline functions and let s be any member of $S(4, P)$ satisfying the condition:

$$\int_{x_{i-1}}^{x_i} [f(x)-s(x)]dx=0, \qquad i=1,2,\cdots,n. \tag{2.1}$$

A convenient representation for a member s in $S(4, P)$ is given, for $x_{i-1}\le x\le x_i$, by

$$6ps(x)=M_{i-1}(x_i-x)^3+M_i(x-x_{i-1})^3+6c_i(x_i-x)+6d_i(x-x_{i-1}), \qquad i=1,2,\cdots,n, \tag{2.2}$$

where $M_i=s''(x_i)$ and c_i and d_i are arbitrary constants to be determined. Since s satisfies the condition (2.1), in view of smoothness requirements of s it is easy to see that

$$c_i=d_{i-1}, \tag{2.3}$$

*) This work was supported by U. G. C.-F. No. 26-1 (721)/90 (SR-IV).

(2.4) $$p^2M_i=d_{i-1}-2d_i+d_{i+1} \quad \text{and}$$
(2.5) $$24F_i=(M_{i-1}+M_i)p^3+12(d_{i-1}+d_i)p$$
where
$$F_i=\int_{x_{i-1}}^{x_i} f(x)dx.$$

If s is also 1-periodic, then elimination of d_i between (2.4) and (2.5), yields:

(2.6) $$M_{i-2}+11M_{i-1}+11M_i+M_{i+1}=(24/p^3)(F_{i-1}-2F_i+F_{i+1}), \quad i=1,2,\cdots,n,$$

where $M_0=M_n$, $M_{-1}=M_{n-1}$, $F_0=F_n$ and $F_{n+1}=F_1$. In fact, we have the recurrence relation

(2.7) $$m_i+m_{i-1}=g_i-g_{i-1}$$
where
$$m_i=M_{i-1}+10M_i+M_{i+1}, \quad \text{and}$$
(2.8) $$g_i=(24/p^3)(F_{i+1}-F_i), \qquad i=1,2,\cdots,n.$$

Therefore if we assume that n is odd then multiplying (2.7) by $(-1)^i$ and summing up we get following system of n equations to determine n unknowns M_i's:

(2.9) $$M_{i-1}+10M_i+M_{i+1}=(12/p^3)\sum_{k=0}^{n-1}(-1)^k[F_{k+i+2}-2F_{k+i+1}+F_{k+i}], \quad i=1,2,\cdots,n.$$

Clearly the coefficient matrix of above system of equations is diagonally dominant and is thus invertible. Therefore a unique solution to the system of equations (2.9) exists and each M_i is determined uniquely. Constants c_i and d_i are then determined correspondingly.

Clearly, when n is even, equation (2.6) does not admit any nontrivial solution for M_i's.

We have thus proved the following:

Theorem 2.1. *For a 1-periodic locally integrable function f defined over $[0,1]$, there exists a unique 1-periodic cubic spline $s\in S(4,P)$ satisfying the AMC (2.1) if and only if n is odd.*

This theorem covers the case $\mu(x)=x$, not considered in [3]. However, the existence of such an interpolatory spline can also be established by an application of Theorem 1 of [1].

3. Error-estimates. In this section we aim to obtain the error-bounds for the spline interpolant of Theorem 2.1. We shall denote $s^{(r)}-f^{(r)}$ by $e^{(r)}$; $r=0,1,2$ and $g^{(r)}(x_i)=g_i^{(r)}$ for any function g defined in $[0,1]$. Thus (2.9) can be written as

(3.1) $$e''_{i-1}+10e''_i+e''_{i+1}=(f''_i-f''_{i-1})+(f''_i-f''_{i+1})+12f''_i + (12/p^3)\sum_{k=0}^{n-1}(-1)^k(F_{k+i+2}-2F_{k+i+1}+F_{k+i}).$$

Now applying Taylor's Theorem and its dual form we observe that

(3.2) $$F_{i+1}-2F_i+F_{i-1}=p^3f''(\phi_i)+(p^3/6)[f''(\theta_{i+1})-f''(\theta_i) -(f''(\Omega_i)-f''(\Omega_{i-1}))]$$

where ϕ_i, θ_i, Ω_i lie in (x_{i-1}, x_i).

Therefore in view of (3.1) and (3.2) we have

$$8|e_i''|\leq 14w(f'', p)+12[(n-1)/2+n/3]w(f'', 2p).$$

Thus, $\quad |e_i''|\leq(7/4)w(f'', p)+[(5-3p)/2]\|f'''\|,$

where $w(g, \delta)=\sup\limits_{|x-y|<\delta}|g(x)-g(y)|$ for $g(x)\in C[0,1]$ $\delta>0$, and $\|\ \|$ means norm in $L^{\infty}[0,1]$.

Since s'' is linear between the mesh points, we have

$$\|e''\|\leq(11/4)w(f'', p)+[(5-3p)/2]\|f'''\|.$$

Thus we have established the following

Theorem 3.1. *If f is a 1-periodic function in $C^3[0,1]$ and s is its 1-periodic cubic spline interpolant satisfying the condition* (2.1), *then*

$$|e_i''|\leq(7/4)w(f'', p)+[(5-3p)/2]\|f'''\|,$$

and

$$\|e''\|\leq(11/4)w(f'', p)+[(5-3p)/2]\|f'''\|.$$

Remark 3.1. The above theorem asserts that $e''(x)$ is uniformly bounded for all n, so that we have

$$e(x)=O(p^2), \qquad e'(x)=O(p).$$

Acknowledgement. The author is grateful to Prof. H.P. Dikshit for some valuable suggestions.

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