58. Itô's Formula for Pseudo-processes

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In [5] there have been introduced the so-called "pseudo-processes" and a stochastic calculus has been developed (in a certain direction). In [7], starting with an equivalent definition for pseudo-processes, a stochastic integral has been introduced and afterwards it has been developed a similar calculus but in other directions. What was missing in [7] is an Itô formula for the stochastic integral, which will be discussed in this article.

Let W be a nonvoid set. We shall denote by Fin $[0,1]$ the set of all finite subsets for the real interval $[0,1]$. We suppose that for every $I \in$ Fin $[0,1]$ there is given $\mathcal{F}_I = a$ tribe of parts from W such that $(\mathcal{F}_I)_{I \in \mathrm{Fin}[0,1]}$ is an increasing family. We denote by $\mathcal{A}_t = \{\cup \mathcal{F}_I\,;\, I \in \mathrm{Fin}\,[0,t]\}$ and $\mathcal{F}_t =$ the tribe generated by $\mathcal{A}_t$. It is considered to be given $P: \mathcal{A}_1 \to \boldsymbol{R}_+$ additive such that $P^I := P|_{\mathcal{F}_I}$ is a probability measure for every $I \in \mathrm{Fin}\,[0,1]$. Let $t \in [0,1]$ be fixed; by $L^1(W, \mathcal{A}_t, P)$ we understand the set of all $H: \mathcal{A}_1 \to \boldsymbol{R}$ additive such that, for every $I \in \mathrm{Fin}\,[0,1]$, $H^I = H|_{\mathcal{F}_I}$ is a real measure, H^I is absolutely continuous with respect to P and the Radon-Nikodym derivative $\dfrac{dH^I}{dP}$ is $\mathcal{F}_t$-measurable a P^I integrable. When the pseudoprocess H_t has (locally) bounded densities and X is a usual martingale (or a semimartingale), then the stochastic integral $(H \otimes X)^I_t$ is given by the formula $\int_0^t \dfrac{dH^I_s}{dP} dM_s + \int_0^t \dfrac{dH^I_s}{dP} dA_s$ where the first is an Itô-type integral (with respect to a local martingale), the second is Stieltjes and $X = M + A$. For example (see [7]), if H_t is a pseudo-martingale, then $H \otimes X$ is well-defined and the result is an $\mathcal{F}_I \cap \mathcal{F}_t$-usual martingale. Hence, if $f \in C^2(R)$ then, by Itô, $f(H \otimes X)$ is a semimartingale.

In the sequel, the stochastic integral is made with respect to usual Brownian motion $B_t (0 \leq t \leq 1)$ and the integrand admits bounded densities or $\int \left(\dfrac{dH^I_s}{dP}\right)^2 ds < +\infty$. Both imply that the stochastic integral is will defined. We have the following Itô formula

$$f((H\otimes B)^I_t) = f((H\otimes B)^I_0 + (f'((H\otimes B)_-) \cdot H\otimes B)^I_t$$
$$+ \frac{1}{2}\int_0^t f''(H\otimes B)^I_{s-} \left(\frac{dH^I_s}{dP}\right)^2 ds.$$

First remark that the integrals are well-defined: the densities of the first integral are bounded and for the second, we can use Schwarz inequality. Secondly, for the proof, it is sufficient to prove the formula for polynomials, hence for the product fg (if the formula is true for f and g separately). In particular $H\otimes B=((H\otimes B)\cdot H)\otimes B$.

An extension of this formula can be obtained by taking as B_t not a Brownian motion but a solution of Meyer's equation $[B]_t=t+\int_0^t g(B_{s-})dB_s$ (see [4]) with $g\neq 0$ (for $g=0$ we obtain $[B]_t=t$ so $[B]_t$ is continuous hence B_t is. That is the Brownian motion). The result is similar to that of Emery's [1].

Combining this formula with the explicit equation solved in [7], Prop. 5, one obtains that the measure $H_t(A)=\int_A \frac{dH_0}{dP}\exp(B_t)dP$ is the unique solution of the equation

$$f(H_t)=f(H_0)+f((H\otimes B)_0)+f'((H\otimes B)_-\cdot H)\otimes B)_t$$
$$+\frac{1}{2}\int_0^t f''(H\otimes B)_{s-}\left(\frac{dH_s^I}{dP}\right)^2 ds,$$

for every f like in Itô's formula.

It is the moment to remark that, if the pseudo-process H satisfies certain conditions, then the stochastic integral $H\otimes B$ represents a good candidate for a "Fölmer measure". More precisely, define $m_{H\otimes X}^I((s,t]\times A)=\int_A [(H\otimes X)_t^I-(H\otimes X)_s^I]dP$ for $0\leq s\leq t\leq 1$ and $A\in\mathcal{A}_s(=\cup_I\mathcal{F}_I,\ I\subset[0,s]$ is finite). The fact that it is well defined can be seen in [5] or [6]. The result is the following: if the pseudo-process H admits positive densities and X is a positive martingale continuous in mean (in particular for a Brownian motion or Meyer's solution of the structure equations), then $m_{H\otimes X}^I$ is countably additive. Indeed, one can see that $m_{H\otimes X}^I((s,t]\times A)=\iint_s^t \frac{dH_r^I}{dP}dX_r dP$; the positivity of the densities of H implies that m is positive. To obtain the desired result, use Kluvanek criterion [3]:

(a) $$\lim_{s\to t} m_{H\otimes X}^I((0,s]\times W)=m_{H\otimes X}^I((0,t]\times W) \quad \text{and}$$

(b) $$\lim_{n\to+\infty}\sup\{m_{H\otimes X}^I(C);\ C\subset(0,1]\times A_n\}=0 \qquad \text{for } \mathcal{A}_1\ni A_n\searrow\varnothing.$$

To verify (a) observe $\iint_0^s \frac{dH_r^I}{dP}dX_r dP$: the inside integral is of Itô-type, hence it converges to $\int_0^t\cdots$ but X is mean continuous, and so is the inside integral; this implies the convergence of (a). For the second statement, take $T(\omega)=\inf\{t;\ (t,\omega)\in C\}$ and, using the fact that $\mathcal{A}_1$ is generated by the stochastic intervals of stopping times, we obtain

$$m_{H\otimes X}^I(C)\leq m_{H\otimes X}^I((T,1])=\iint_{(T,1]}\frac{dH_r^I}{dP}dX_r dP\leq\int_{A_n}(H\otimes X)_1^I dP\longrightarrow 0$$

by Beppo-Levi ($A_n\searrow\varnothing$).

Now, as a process, what is the shape of $(H\otimes X)_t^I$? Using the technique used in [2], we shall prove that, under a suitable hypothesis, the stochastic integral is a quasi-martingale. Indeed, from the existence of

$(H\otimes X)^I$ we see that the densities are locally bounded and from the existence of $m^I_{H\otimes X}$ it follows that the densities are positive. So, if the densities of H are uniformly bounded and positive, then the variation (as a measure) of $m^I_{H\otimes X}$ is bounded; hence by [2] we obtain that the stochastic integral is a quasi-martingale. By the present Itô formula, we obtain that for any $f\in C^2(R)$, the process $f(H\otimes B)_t$ is a quasi-martingale.

Moreover, even in the vectorial case (when the processes take values in a reflexive Banach space), the conclusions remain true as is seen by understanding the integrability in the sense of Pettis (see [8]). By the way, it is sufficient to consider the vectorial case only; the scalar case being its consequence for positive processes (the converse is generally false, as it is shown in [8]).

As a particular (vectorial) case, consider that the densities take values in the Banach space of linear and continuous function from a Banach space G into a Hilbert space E and the trajectories are P-continuous (a.e.). We integrate H with respect to $B_t(0\le t\le 1)$ behaving like a Brownian motion, each B_t being G-valued random variable (in short: $W-G$ r.v.) (see [8]). Denote by $(H\otimes B)^I_t$ (after McShane) the limit in probability of $\sum_{j=1}^n \frac{dH^I_{s_j}}{dP}$ $(B_{t_{j+1}}-B_{t_j})$ when $\max_{j=1,\dots,n}(t_{j+1}-t_j)\to 0$ if $0=t_1\le t_2\le\cdots t_{n+1}=t\le 1$ and $s_j\le t_j$. The integral is well-defined (see [8]) if, for example $\sup_{0\le t\le 1}\left\|\frac{dH_t}{dP}\right\|$ $\le$ const. P-a.e. and $\left\|\frac{dH^I_1}{dP}\right\|$ is uniformly P-integrable. One can easily see that, if the densities are vectorial quasi-martingales, then the integral is a vectorial quasi-martingale, too. For such integrals, Itô formula is similar to the one at the beginning (with slight modifications: f is twice Fréchet differentiable on the space of r.v. from W to E and the last integral contains $\left\|\frac{dH^I_s}{dP}\right\|^2$ (the norm of G-E r.v.). One can make similar remarks concerning this formula and the equations solved in [8].)

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