21. The Godbillon-Vey Invariant and the Foliated Cobordism Group*)

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(Communicated by Shokichi IYANAGA, M. J. A., April 13, 1992)

Introduction. In this paper we show the following statement: Let $\mathcal{F}$ be a codimension-1 transversely oriented foliation of a closed oriented 3-manifold M. The Godbillon-Vey invariant of $\mathcal{F}$ is zero if and only if $\mathcal{F}$ is foliated cobordant to a codimension-1 transversely oriented foliation $\mathcal{G}$ of a closed oriented 3-manifold N and there exists a sequence $\mathcal{G}_k$ of null-cobordant codimension-1 foliations of N converging to $\mathcal{G}$.

Two codimension-1 transversely oriented foliations $(M, \mathcal{F})$ and $(N, \mathcal{G})$ of closed oriented n-manifolds are *foliated cobordant* if there exists a codimension-1 transversely oriented foliation $(W, \mathcal{H})$ of a compact oriented $(n+1)$-manifold such that $\partial W = (-M) \cup N$, $\mathcal{H}$ is transverse to ∂W and the restrictions $\mathcal{H}|M$ and $\mathcal{H}|N$ coincide with $\mathcal{F}$ and $\mathcal{G}$, respectively. The foliated cobordism classes form an additive group $\mathcal{F}\Omega_{n,1}$. The foliations $(M, \mathcal{F})$ representing the zero of the foliated cobordism group are those cobordant to the empty set. We say they are *null-cobordant*.

The Godbillon-Vey invariant for a codimension-1 transversely oriented foliation $\mathcal{F}$ was defined as follows ([7]). Let ω be a 1-form defining $\mathcal{F}$. The integrability condition is the existence of 1-form η such that $d\omega = \omega \wedge \eta$. Then the 3-form $\eta \wedge d\eta$ is closed and its cohomology class depends only on the foliation $\mathcal{F}$. If $\mathcal{F}$ is a codimension-1 transversely oriented foliation of a closed oriented 3-manifold M, then the Godbillon-Vey invariant is the integral of this 3-form.

There are two properties which follow easily from the definition ([7]). One is that this invariant depends only on the cobordism class of the foliations. This is an easy consequence of the Stokes theorem. The other is that this invariant varies continuously when we deform the foliation. The reason is that the 1-form η can be taken to be the Lie derivative $L_X\omega$, where X is a vector field such that $\omega(X) = 1$. The examples for these continuous variations were given by Thurston ([15]), and hence we have a surjective homomorphism $GV: \mathcal{F}\Omega_{3,1} \to \boldsymbol{R}$. The natural question on the injectivity is still an open question.

We can ask a weaker question. By the property of continuous variation of GV, if a foliation is approximated by null-cobordant foliations, its GV is zero. Moreover, if a foliation is cobordant to such an approximable foliation, then its GV is zero. Now the weaker question is whether the

*) This paper is dedicated to the memory of Itiro Tamura (1926–1991).

converse is true. The above statement says the converse holds.

To give the precise statement and prove it, we need to enlarge the domain of definition of the Godbillon-Vey invariant ([24]). We review it in § 1, and we give the precise statement of our main theorem. The proof of our main theorem relies on a study of the group of piecewise linear (PL) homeomorphisms of the real line in [26], which we review in § 2. We give the proof of our main theorem in § 3.

§ 1. Main theorem. First we give the domain of definition of the Godbillon-Vey invariant which we consider in this paper ([24]).

A foliated $\boldsymbol{R}$-product with compact support over a suface Σ is a foliation of the product $\Sigma \times \boldsymbol{R}$ transverse to the fibers of the projection $\Sigma \times \boldsymbol{R} \rightarrow \Sigma$ which coincides with the product foliation with leaves $\Sigma \times \{*\}$ outside a compact set. (Foliated S^1-products are defined similarly.) By considering these foliated $\boldsymbol{R}$-products with compact support over surfaces, the Godbillon-Vey invariant gives rise to a 2-cocycle of the group of diffeomorphims of $\boldsymbol{R}$ with compact support ([1]).

We formulate the domain of definition of the Godbillon-Vey 2-cocycle.

Let β be a real number not less than 1. For a function φ on $\boldsymbol{R}$ with compact support, we put $V_\beta(\varphi) = \sup \sum_{j=1}^{k} |\varphi(x_j) - \varphi(x_{j-1})|^\beta$, where the supremum is taken over all finite subsets $\{x_0, \cdots, x_k\}$ $(x_0 < \cdots < x_k)$ of $\boldsymbol{R}$. We call it the β-variation of φ. The functions on $\boldsymbol{R}$ with compact support whose β-variations are bounded form a normed linear space $\mathcal{V}_\beta$ with respect to the following β-norm $|||\ \ |||_\beta$: $|||\varphi|||_\beta = V_\beta(\varphi)^{1/\beta}$.

Let $\boldsymbol{G}_c^{L,\mathcal{V}_\beta}(\boldsymbol{R})$ be the group of Lipschitz homeomorphisms f with compact support such that $\log f'(x-0)$ exist as elements of $\mathcal{V}_\beta$. This $\boldsymbol{G}_c^{L,\mathcal{V}_\beta}(\boldsymbol{R})$ contains the group $PL_c(\boldsymbol{R})$ of PL homeomorphisms of $\boldsymbol{R}$ with compact support as well as the group $\boldsymbol{G}_c^{1+1/\beta}(\boldsymbol{R})$ of diffeomorphisms of class $C^{1+1/\beta}$ of $\boldsymbol{R}$ with compact support.

We have the following proposition ([24]).

Proposition 1.1. *$\boldsymbol{G}_c^{L,\mathcal{V}_\beta}(\boldsymbol{R})$ $(1 \leq \beta)$ has the following right invariant metric: For f_1 and f_2 of $\boldsymbol{G}_c^{L,\mathcal{V}_\beta}(\boldsymbol{R})$ $(1 \leq \beta)$, $\mathrm{dist}(f_1, f_2) = |||\log(f_1 \circ f_2^{-1})'(x-0)|||_\beta$. There is a 2-cocycle GV for the group $\boldsymbol{G}_c^{L,\mathcal{V}_\beta}(\boldsymbol{R})$ $(1 \leq \beta \leq 2)$ which is an extension of the Godbillon-Vey cocycle, and $(f_1 \circ f_2, f_2) \mapsto GV(f_1, f_2)$ is continuous with respect to the above metric.*

$GV(f_1, f_2)$ is in fact the area enclosed by the curve $(\log(f_1 \circ f_2)', \log f_2')$ in the Euclidean plane ([1], [10], [24]).

We also consider a similar group $\boldsymbol{G}^{L,\mathcal{V}_\beta}(S^1)$ $(1 \leq \beta)$ as well as the groupoid $\Gamma_1^{L,\mathcal{V}_\beta}$ of germs of such homeomorphisms. We say a foliation is of class $C^{L,\mathcal{V}_\beta}$ if it is a $\Gamma_1^{L,\mathcal{V}_\beta}$-structure with local projections being smooth submersions ([9]).

Thus we can consider the family of foliations of class $C^{L,\mathcal{V}_\beta}$ with $1 \leq \beta < 2$ as a domain of definition of the Godbillon-Vey invariant. Note that the family of foliations of class $C^{L,\mathcal{V}_\beta}$ $(1 \leq \beta < 2)$ contains the foliations of class $C^{1+1/\beta}$ $(1/\beta > 1/2)$ where Hurder and Katok defined the Godbillon-Vey invariant ([10]) as well as the transversely PL foliations where Ghys and Sergiescu defined the discrete Godbillon-Vey invariant ([6], [4]). The definition of the 2-cocycle in Proposition 1.1 is an extension of both of these. This domain of definition is almost optimal ([23], see also [22], [25]).

Now we can state our theorem.

Theorem 1.2. *Let $\mathcal{F}$ be a codimension-1 transversely oriented foliation of class $C^{1+\alpha}$ $(1/2<\alpha\leq 1)$ of a closed oriented 3-manifold M. The Godbillon-Vey invariant of $\mathcal{F}$ is zero if and only if $\mathcal{F}$ is foliated cobordant to a codimension-1 transversely oriented foliation $\mathcal{G}$ of class $C^{1+\alpha}$ of a closed oriented 3-manifold N and there exists a sequence $\mathcal{G}_k$ of codimension-1 null-cobordant foliations of class $C^{L,\mathcal{V}_{1/\alpha}}$ of N converging to $\mathcal{G}$ in the $C^{L,\mathcal{V}_\beta}$ topology $(1/\alpha<\beta<2)$.*

As we will see in the proof, $\mathcal{G}$ is a foliated S^1-product over a surface Σ and the meaning of convergence is that for any $\gamma\in\pi_1(\Sigma)$, the holonomy along γ converges.

We have the following generalization of Theorem 1.2.

Theorem 1.3. *Let $\mathcal{F}$ be a codimension-1 transversely oriented foliation of class $C^{L,\mathcal{V}_\beta}$ of a closed oriented 3-manifold M. The Godbillon-Vey invariant of $\mathcal{F}$ is zero if and only if $\mathcal{F}$ is foliated cobordant to a codimension-1 transversely oriented foliation $\mathcal{G}$ of class $C^{L,\mathcal{V}_\beta}$ of a closed oriented 3-manifold N and there exists a sequence $\mathcal{G}_k$ of codimension-1 null-cobordant foliations of class $C^{L,\mathcal{V}_\beta}$ of N converging to $\mathcal{G}$ in the $C^{L,\mathcal{V}_{\beta'}}$ topology $(\beta<\beta'<2)$.*

In Theorem 1.3, $\mathcal{G}$ is a foliated S^1-product over a surface Σ as before. However, we should be careful about meaning of the convergence because $\boldsymbol{G}^{L,\mathcal{V}_\beta}(S^1)$ is not a topological group. The convergence means that after fixing a triangulation with one vertex of Σ, the holonomy along any edge converges. In Theorem 1.2 we did not meet such difficulty because the composition and the inversion of $\boldsymbol{G}^{L,\mathcal{V}_\beta}(S^1)$ are continuous at the elements of $\boldsymbol{G}^{1+1/\beta}(S^1)$.

To obtain our main theorems, we need to approximate a foliation by foliations which we can control their cobordism classes. We use the transversely PL foliations which are investigated by Greenberg ([8]) and Ghys-Sergiescu ([6]) (see also [27]). Though transversely PL foliations are not smooth foliations, they are in our domain of definition of the Godbillon-Vey invariant and the invariant varies continuously with respect to the topology introduced above.

Remark. There are PL foliations defined by Gel'fand and Fuks ([3]). They are related to the above transversely PL foliations but they are different from them.

§ 2. The group of piecewise linear homeomorphisms. We study the PL-foliated $\boldsymbol{R}$-products over surfaces. It turns out to be important to write a PL homeomorphism of $\boldsymbol{R}$ with compact support close to the identity as a product of a fixed number of commutators of PL homeomorphisms of

$\boldsymbol{R}$ close to the identity. Then we get an information on the second homology of the group $PL_c(\boldsymbol{R})$ of PL homeomorphisms of $\boldsymbol{R}$ with compact support. We proved the following theorems in [26]. A PL homeomorphism of $\boldsymbol{R}$ with compact support is said to be *elementary* if it has at most 3 nondifferentiable points.

Theorem 2.1. *Let β be a real number not less than* 1. *There exist positive real numbers c and C satisfying the following conditions. Let ε be a positive real number such that $\varepsilon \leq c$. Let f be an elementary PL homeomorphism of $\boldsymbol{R}$ with support in* [1/8, 7/8]. *Assume that $|||\log f'|||_\beta \leq \varepsilon^2$. Then f is written as a product* (*composition*) *of* 3 *commutators of PL homeomorphisms of $\boldsymbol{R}$ as follows: $f=[g_1, g_2][g_3, g_4][g_5, g_6]$, where the supports of g_i $(i=1, \cdots, 6)$ are contained in* [0, 1] *and $|||\log g_i'|||_\beta \leq C\varepsilon$.*

Theorem 2.2. *Let β be a real number not less than* 1. *Let q be a positive integer and δ, a positive real number. There exist positive real numbers c and C satisfying the following conditions. Let ε be a positive real number such that $\varepsilon \leq c$. Let f be a PL homeomorphism of $\boldsymbol{R}$ with support in* [1/4, 3/4] *such that the number of the nondifferentiable points of f is at most $4\varepsilon^{-q}+2$ and $|||\log f'|||_\beta \leq \delta\varepsilon^3$. Then $f=\prod_{i=1}^{16(q+1)}[g_{2i-1}, g_{2i}]$, where the supports of g_j $(j=1, \cdots, 32(q+1))$ are contained in* [0, 1] *and $|||\log (g_j)'|||_\beta \leq C\varepsilon$.*

Using Theorem 2.1 and a construction which is a combination of those in [18] and in [20], we showed the following theorem in [25].

Theorem 2.3. *Let a, b, a', b' be real numbers such that $ab=a'b'$. Let f_a and $f_{a'}$ be PL homeomorphisms of $\boldsymbol{R}$ with support in* $[-1, 0]$ *such that $\log f_a'(-0)=a$ and $\log f_{a'}'(-0)=a'$, respectively, and let g_b and $g_{b'}$ be PL homeomorphisms of $\boldsymbol{R}$ with support in* [0, 1] *such that $\log g_b'(+0)=b$ and $\log g_{b'}'(+0)=b'$, respectively. Then the* 2*-cycles $(f_a, g_b)-(g_b, f_a)$ and $(f_{a'}, g_{b'})-(g_{b'}, f_{a'})$ are homologous in $BG^{L,\mathcal{V}_\beta}$ $(\beta \geq 1)$.*

Now the foliated cobordism group $\mathcal{F}\Omega_{3,1}^{PL}$ of transversely PL foliations is isomorphic to $H_2(BPL_c(\boldsymbol{R}); \boldsymbol{Z})$ (see for example [21]). Greenberg ([8]) showed that $\mathcal{F}\Omega_{3,1}^{PL}$ is generated by the PL Reeb foliations of S^3 which is defined as follows. Consider the foliation of $\boldsymbol{R}^2 \times [0, \infty)$ by planes $\boldsymbol{R}^2 \times \{*\}$. This foliation is invariant under the similarity transformation with center (0, 0, 0) and with ratio e^a and the foliation induces a foliation of the solid torus $(\boldsymbol{R}^2 \times [0, \infty)-(0, 0, 0))/(x, y, z) \sim e^a(x, y, z)$. By attaching two such foliated solid tori, we obtain a PL Reeb foliation of S^3. The PL Reeb foliation of S^3 whose compact toral leaf has the germs at 0 of f_a and g_b above as holonomies is mapped to the class of $(f_a, g_b)-(g_b, f_a)$ in $H_2(BPL_c(\boldsymbol{R}); \boldsymbol{Z})$ by the isomorphism. Since the (discrete) Godbillon-Vey invariant of such foliation is equal to ab ([6]), we have the following corollary.

Corollary 2.4. *The foliated cobordism class as foliations of class $C^{L,\mathcal{V}_\beta}$ $(1 \leq \beta < 2)$ of transversely oriented transversely PL foliations of closed oriented* 3*-manifolds is characterized by its* (*discrete*) *Godbillon-Vey class.*

As to the approximation of a foliation by PL foliations, the following stable approximation theorems are obtained as an application of Theorem 2.2 ([26]).

Theorem 2.5. *Let $\mathcal{G}$ be a foliated $\boldsymbol{R}$-product of class $C^{1+\alpha}$ with support in $[1/4, 3/4]$ over the closed oriented surface Σ_N of genus N. Let β be a positive real number greater than $1/\alpha$. Then there are a positive integer M and a family of PL-foliated $\boldsymbol{R}$-products $\mathcal{G}_k$ over the connected sum $\Sigma_N \# \Sigma_M$ such that $\mathcal{G}_k$ converges to $\mathcal{G} \star \mathcal{P}$ in the $C^{L,\mathcal{V}_\beta}$ topology, where $\mathcal{P}$ be the trivial foliated $\boldsymbol{R}$-product over Σ_M. In particular, if $1/\alpha<\beta<2$, the Godbillon-Vey invariant $GV(\mathcal{G}_k)$ converges to $GV(\mathcal{G})$.*

Theorem 2.6. *Let $\mathcal{G}$ be a foliated $\boldsymbol{R}$-product of class $C^{L,\mathcal{V}_\beta}$ with support in $[1/4, 3/4]$ over a closed oriented surface Σ_N of genus N with a triangulation with one vertex. Let β' be a positive real number greater than β. Then there exist a positive integer M, a closed oriented surface Σ_{N+M} of genus $N+M$ with a triangulation with one vertex, a simplicial map $s: \Sigma_{N+M} \to \Sigma_N$ of degree 1 and a family of PL-foliated $\boldsymbol{R}$-products $\mathcal{G}_k$ over Σ_{N+M} such that $\mathcal{G}_k$ converges to the induced foliated product $s^*\mathcal{G}$ in the $C^{L,\mathcal{V}_{\beta'}}$ topology. In particular, if $\beta<\beta'<2$, the Godbillon-Vey invariant $GV(\mathcal{G}_k)$ converges to $GV(\mathcal{G})$.*

We have the following corollary to these theorems.

Corollary 2.7. *If the Godbillon-Vey invariant of the foliated $\boldsymbol{R}$-product $\mathcal{G}$ in Theorems 2.5 or 2.6 is zero, then $\mathcal{G}_k$ can be taken so that their Godbillon-Vey invariants are zero.*

§ 3. Proof of the main theorem. Let $\mathcal{F}$ be a transversely oriented foliation of a closed oriented manifold M of class $C^{1+\alpha}$ ($\alpha \geq 1/2$) such that the Godbillon-Vey invariant is zero. First we use the following theorem.

Theorem 3.1. *Any codimention-1 transversely oriented foliation is cobordant to a foliated S^1-product over the closed oriented surface Σ_2 of genus 2.*

This is shown by using a theorem of Mather ([12], [16], [13], [14]) and the theorem of existence of foliations of Thurston ([17]). (See also [18], [21].) The foliated S^1-product over Σ_2 can be taken so that the foliation coincides with the product foliation on $\Sigma_2 \times [1/2, 1] \subset \Sigma_2 \times (\boldsymbol{R}/\boldsymbol{Z})$.

Using Theorem 3.1, we obtain a $C^{1+\alpha}$-foliated S^1-product $\mathcal{G}'$ over Σ_2. Let β be a real number such that $1/\alpha<\beta<2$. By Theorem 2.5 and Corollary 2.7, there exist an integer M and a family of PL-foliated $\boldsymbol{R}$-products $\mathcal{G}_k$ over the connected sum $\Sigma_2 \# \Sigma_M$ such that the Godbillon-Vey invariant of $\mathcal{G}_k$ is zero and the sequence $\mathcal{G}_k$ converges to $\mathcal{G} \star \mathcal{P}$ in the $C^{L,\mathcal{V}_\beta}$ topology. Then Corollary 2.4 assures that $\mathcal{G}_k$ are null-cobordant as foliations of class $C^{L,\mathcal{V}_\beta}$. Thus we proved Theorem 1.2.

Theorem 1.3 is shown in the same way except that we use Theorem 2.6 instead of Theorem 2.5.

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