# 48. A Class of Norms on the Spaces of Schwarzian Derivatives and its Applications*)

By Toshiyuki SUGAWA

Department of Mathematics, Kyoto University

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**§0. Introduction.** As is well-known, the hyperbolic-sup norm (or the Nehari norm) of the Schwarzian derivative of a meromorphic function is closely related to its (global or local) univalence. The famous Nehari-Kraus theorem and Ahlfors-Weill theorem are of fundamental importance in this direction of research.

In this note, in order to clarify this relationship more, we shall introduce, in section 2, a class of "local" norms on the space of Schwarzians. These norms are expected to be near the hyperbolic-sup norm, and determined by the local shape of the domain. But, whereas the pullback by a conformal map is an isometry with the hyperbolic-sup norm, it is only a quasi-isometry with these local norms. In section 3, we shall describe how the magnitude of norms of Schwarzian is controled by the local quasiconformal(= qc) extensibility, which the author has learned from [1]. An essential use of the result in this section will be made in the article [5] of the author. Finally, in section 4, we shall mention an estimate of the local norms of Schwarzian by the injectivity radius.

**§1. Preliminaries.** Throughout this note, let $D$ be a plane domain of hyperbolic type (i.e., $\boldsymbol{C}\setminus D$ contains at least two points) and $\rho_D(z)\,|\,dz\,|$ be the hyperbolic metric with constant negative curvature $-4$. For a holomorphic function $\varphi$ on $D$, we define the hyperbolic-sup norm of $\varphi$ by $\|\varphi\|_D = \sup_{z\in D}\rho_D(z)^{-2}\,|\,\varphi(z)\,|$ and we denote by $B_2(D)$ the space of all holomorphic functions in $D$ with a finite norm, which becomes a complex Banach space. For a non-constant meromorphic function $f$ on $D$, the Schwarzian derivative of $f$ is defined by the formula $S_f = (f''/f')' - \dfrac{1}{2}(f''/f')^2$, which is holomorphic at $z_0 \in D$ if and only if $f$ is locally univalent at $z_0$.

In this note, $f : \hat{\boldsymbol{C}} \to \hat{\boldsymbol{C}}$ shall be called a *k-qc map* of $\hat{\boldsymbol{C}}$ where $k$ is a constant and $0 \le k < 1$, if $f$ is an orientation-preserving self-homeomorphism of the Riemann sphere $\hat{\boldsymbol{C}}$ with locally $L^2$-derivatives such that $|\,\partial_{\bar z} f\,| \le k\,|\,\partial_z f\,|$ a.e. It should be alerted that this terminology is not standard. In fact, $k$-qc map is ordinarily called "$K$-qc" where $K = \dfrac{1+k}{1-k}$. As a general reference for qc maps and the hyperbolic sup-norm of the Schwarzian derivatives, we refer to [4].

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The following theorem is fundamental for our present aim (e.g. see [4] pp. 60, 72 and 87). The first assertion and the last one are known as the Nehari-Kraus theorem and the Ahlfors-Weill theorem, respectively.

**Theorem 1.1.** *If a meromorphic function on the disk $\Delta$ is univalent, then $\| S_f \|_\Delta \le 6$. Moreover if $f$ is extended to a $k$-qc map of $\hat{\boldsymbol{C}}$, then $\| S_f \|_\Delta \le 6k$.*

*Conversely, each meromorphic function $f$ on a disk $\Delta$ with $\| S_f \|_\Delta \le 2$ is univalent, and if $\| S_f \|_\Delta < 2$, then $f$ can be extended to a $\frac{1}{2}\| S_f \|_\Delta$-qc map of $\hat{\boldsymbol{C}}$.*

**§2. A class of norms.** Now we define certain norms which are determined by only local deta of the domain. The same ideas here were appeared in some earlier works in this area, not necessarily in explicit forms.

Let $1 \le A < \infty$ be a constant, $D$ be a plane domain and define $\mathfrak{D}_A(D) = \{B(z_0, r)\, ;\, r > 0,\ B(z_0, Ar) \subset D\}$, where $B(z_0, r) = \{z \in \boldsymbol{C}\, ;\, |z - z_0| < r\}$. For a holomorphic function $\varphi$, we define

$$\| \varphi \|_D^{(A)} = A^2 \cdot \sup_{\Delta \in \mathfrak{D}_A(D)} \| \varphi \|_\Delta$$

and

$$\| \varphi \|_D^{(\infty)} = \sup_{z \in D} | \varphi(z) | \operatorname{dist}(z, \partial D)^2.$$

The reason for the above notation $\| \cdot \|_D^{(\infty)}$ will be explained in Remark 1 of Theorem 2.1.

The hyperbolic-sup norm have a monotonicity property that $\| \varphi \|_{D_1} \le \| \varphi \|_{D_2}$ if $D_1 \subset D_2$, which is a conclusion of the Schwarz-Pick lemma. For the above defined norms this property *trivially* holds, that is, if $D_1 \subset D_2$, then $\| \varphi \|_{D_1}^{(A)} \le \| \varphi \|_{D_2}^{(A)}$ for $1 \le A \le \infty$.

The following theorem gives a basic estimate for our norms.

**Theorem 2.1.**

(a) $\| \varphi \|_D^{(A)} \le \| \varphi \|_D$,

(b) $\| \varphi \|_D^{(A_2)} \le \| \varphi \|_D^{(A_1)}$ *if* $1 \le A_1 \le A_2 \le \infty$,

(c) $\| \varphi \|_D^{(A)} \le \left( \dfrac{2A}{A + \sqrt{A^2 - 1}} \right)^2 \| \varphi \|_D^{(\infty)}$ $(\le 4 \| \varphi \|_D^{(\infty)})$.

*Proof.* Let $1 \le A_1 \le A_2 < \infty$, $\Delta = B(z_0, r) \in \mathfrak{D}_{A_2}(D)$, and set $A = A_2/A_1$, $\Delta' = B(z_0, Ar) (\in \mathfrak{D}_{A_1}(D))$. Since $\rho_\Delta(z) = \dfrac{r}{r^2 - |z - z_0|^2}$, $\rho_{\Delta'}(z) = \dfrac{Ar}{A^2 r^2 - |z - z_0|^2}$, we have $\rho_{\Delta'}(z)/\rho_\Delta(z) \le 1/A$ for all $z \in \Delta$. Thus $\| \varphi \|_\Delta = \sup \rho_\Delta(z)^{-2} | \varphi(z) | \le A^{-2} \sup \rho_{\Delta'}(z)^{-2} | \varphi(z) | \le A^{-2} \| \varphi \|_{\Delta'}$, and this implies $A_2^2 \| \varphi \|_\Delta \le A_1^2 \| \varphi \|_{\Delta'}$. Therefore (b) follows if $A_2 < \infty$. In the case that $A_1 < A_2 = \infty$, for arbitrary $z_0 \in D$ let $\delta = \operatorname{dist}(z_0, \partial D)$, $r = \delta / A_1$ and $\Delta = B(z_0, r)$ $(\in \mathfrak{D}_{A_1}(D))$. Then $\rho_\Delta(z_0) = \dfrac{1}{r} = \dfrac{A_1}{\delta}$, so we have $\delta^2 | \varphi(z_0) | = A_1^2 \rho_\Delta(z_0)^{-2} | \varphi(z_0) | \le A_1^2 \| \varphi \|_\Delta \le \| \varphi \|_D^{(A_1)}$ and thus we now get part (b) for $A_2 = \infty$.

Once (b) obtained, it suffices to prove (a) in the case $A = 1$, in which case (a) naturally follows from the monotonicity of the hyperbolic-sup norm.

Finally we show the statement (c). Let $\delta(z) = \operatorname{dist}(z, \delta D)$ for $z \in D$ and $\Delta = B(z_0, r) \in \mathscr{D}_A(D)$ . Then (c) is directly deduced from the following lower estimate $\delta(z)\rho_\Delta(z) \geq \dfrac{A + \sqrt{A^2 - 1}}{2}$ $(\forall Z \in \Delta)$. Without loss of generality, we can assume that $z_0 = 0$. Since $\delta(z) \geq \delta(0) - |z| \geq rA - |z|$, it follows that $\delta(z)\rho_\Delta(z) \geq \dfrac{r(rA - |z|)}{r^2 - |z|^2}$ and an elementary calculation shows that

$$(*) \qquad \frac{r(rA - |z|)}{r^2 - |z|^2} \geq \frac{A + \sqrt{A^2 - 1}}{2}$$

for $0 \leq |z| < r$, where equality holds for $|z| = \dfrac{r}{A + \sqrt{A^2 - 1}}$.

**Remark 1.** From (b) and (c) it follows that $\| \varphi \|_D^{(\infty)} \leq \| \varphi \|_D^{(A)} \leq \left(\dfrac{2A}{A + \sqrt{A^2 - 1}}\right)^2 \| \varphi \|_D^{(\infty)}$, thus $\lim_{A \to \infty} \| \varphi \|_D^{(A)} = \| \varphi \|_D^{(\infty)}$, which is a reason why we use the notation $\| \cdot \|_D^{(\infty)}$.

**Remark 2.** The constant $\left(\dfrac{2A}{A + \sqrt{A^2 - 1}}\right)^2$ in (c) is best possible. We shall explain this fact for $A > 1$. Let $D = U = B(0,1)$, $r = 1/A$, $\Delta = B(0, r)$ and $a = \dfrac{r}{A + \sqrt{A^2 - 1}}$. Note that $\Delta \in \mathscr{D}_A(D)$ and $a \in \Delta$. Next, we choose a sufficiently large integer $n$ and a positive real number $\alpha \in \left(0, \dfrac{n}{2}\right)$ such that $a = \dfrac{n - 2\alpha}{n + 2}$. Put $\varphi_{n,\alpha}(z) = (z + \alpha)^n$, then $\| \varphi_{n,\alpha} \|_U^{(\infty)} = \sup_{|z|<1}(1 - |z|)^2 |z + \alpha|^n = (1 - a)^2(a + \alpha)^n$. On the other hand, from the equality in $(*)$, we have $\| \varphi_{n,\alpha} \|_\Delta \geq \rho_\Delta(a)^{-2} |a + \alpha|^n = (2/(A + \sqrt{A^2 - 1}))^2 \| \varphi_{n,\alpha} \|_U^{(\infty)}$, hence,

$$\| \varphi_{n,\alpha} \|_U^{(A)} \geq A^2 \| \varphi_{n,\alpha} \|_\Delta \geq \left(\frac{2A}{A + \sqrt{A^2 - 1}}\right)^2 \| \varphi_{n,\alpha} \|_U^{(\infty)}.$$

The opposite inequality is already obtained in Theorem 2.1 (c), and hence we conclude that equality holds in the above.

From the above theorem, it turns out that norms $\| \cdot \|_D^{(A)}$ are equivalent to each other $(1 \leq A \leq \infty)$, so we have a complex Banach space $\tilde{B}_2(D) = \{\varphi : \text{holomorphic function on } D \text{ and } \| \varphi \|_D^{(A)} < \infty\}$, which is independent of the special choice of $A$.

By Theorem 2.1 (a), we obtain that $B_2(D) \subset \tilde{B}_2(D)$, but unfortunately, these two spaces does not coinside generally. The following theorem gives a geometric criterion for the coincidence of the two spaces. (The implication (i) $\Rightarrow$ (ii) is a conclusion from the Banach open mapping theorem.)

**Theorem 2.2** (Beardon-Pommerenke [2]). *The followings are equivalent to each other.*

( i ) $B_2(D) \subset \tilde{B}_2(D)$,

( ii ) *There exists a constant* $c > 0$ *such that* $\| \varphi \|_D \leq c \| \varphi \|_D^{(\infty)}$ *for all* $\varphi \in \tilde{B}_2(D)$,

(iii) $\sup\{\operatorname{mod} A\,;A$ *is an annulus in* $D$ *which separates the boundary of* $D\} < \infty$, *where* $\operatorname{mod} A = \dfrac{\log R}{\log r}$ *if* $A = \{z\,; r < |z - z_0| < R\}$.

On the other hand, for simply connected domains, the Koebe one-quarter theorem yields the following

**Theorem 2.3** (cf. [2]). *If $D$ is a simply connected domain of the hyperbolic type, then*

$$\| \varphi \|_D \leq 16 \| \varphi \|_D^{(\infty)}.$$

Thus, for simply connected plane domains all the above norms are equivalent. We conclude this section with an exposition of the quasi-isometricity of the pullback by conformal maps with respect to these norms.

Suppose that $F$ maps a domain $D$ conformally into $\boldsymbol{C}$. For $\varphi \in B_2(F(D))$ let $F^*\varphi$ denote the pullback of $\varphi$ by $F$ as a holomorphic quadratic differential, that is, $F^*\varphi = \varphi \circ F \cdot (F')^2$. This pullback is an isometry with the hyperbolic-sup norm, that is, $\| F^*\varphi \|_D = \| \varphi \|_{F(D)}$. While, with the local norms, the pullback is only a quasi-isometry.

**Proposition 2.4.** *Suppose that $F$ maps a hyperbolic plane domain $D$ conformally into $\boldsymbol{C}$. Then $\| F^*\varphi \|_D^{(\infty)} \leq 16 \| \varphi \|_{F(D)}^{(\infty)}$. Moreover if $F$ is a Möbius transformation, then we have a better estimate*: $\| F^*\varphi \|_D^{(A)} \leq \left(\dfrac{2}{1 + A^{-2}}\right)^2 \| \varphi \|_{F(D)}^{(A')}$ *where* $A \geq 1$, $A' = \dfrac{A + A^{-1}}{2}$.

*Proof.* The first assertion directly follows from the inequality: $\operatorname{dist}(F(z), \partial F(D)) \geq \dfrac{1}{4} |F'(z)| \operatorname{dist}(z, \partial D)$, which is an easy consequence from the Koebe one-quarter theorem. Next suppose that $F$ is a Möbius map. Let $\Delta = B(z_0, r) \in \mathfrak{D}_A(D)$ and set $\tilde{\Delta} = B(z_0, Ar)$. We now assert that $\Delta' = F(\Delta) \in \mathfrak{D}_{A'}(F(D))$. We may assume that $F(\tilde{\Delta}) = \tilde{\Delta} = U$ and that $F(z) = \dfrac{z + a}{1 + az}$ where $0 \leq a < 1$. Since the center of $F(\Delta)$ is $c = \dfrac{1}{2}(F(A^{-1}) + F(-A^{-1}))$ and the radius is $r = \dfrac{1}{2}(F(A^{-1}) - F(-A^{-1}))$, and therefore $\dfrac{1 - c}{r} \geq A'$, it follows that $F(\Delta) \in \mathfrak{D}_{\frac{1-c}{r}}(U) \subset \mathfrak{D}_{A'}(F(D))$. By the above assertion, we obtain an inequality $\| F^*\varphi \|_\Delta = \| \varphi \|_{F(\Delta)} \leq (A')^{-2} \| \varphi \|_{F(D)}^{(A')}$, which proves the second assertion.

**Remark 1.** In particular, $\| F^*\varphi \|_D^{(1)} = \| \varphi \|_{F(D)}^{(1)}$ if $F$ is a Möbius map.

**Remark 2.** Let $A \geq 1$ and suppose that $F: D \to \boldsymbol{C}$ is a holomorphic map which excludes at least two points and $\| S_F \|_D \leq 2A^2$. Then, by the similar way as above, we can show that $\operatorname{dist}(F(z), \partial F(D)) \geq \dfrac{|F'(z)|}{4A} \operatorname{dist}(z, \partial D)$, therefore we obtain $\| F^*\varphi \|_D^{(\infty)} \leq (4A)^2 \| \varphi \|_{F(D)}^{(\infty)}$.

**§3. The local qc-extensibility.** The "local" norms of the Schwarzian faithfully measure the local qc-extensibility of the function. Now we shall state this in a precise form. Firstly the next lemma directly follows from Theorem 1.1 and the definition of the norm.

**Lemma 3.1.** *Let $D$ be a hyperbolic plane domain, $A \geq 1$ and $k \in [0, 1)$ be constants and $f : D \to \hat{C}$ be a meromorphic function.*

*If $f|_\Delta$ can be extended to a $k$-qc map of $\hat{C}$ for any $\Delta \in \mathfrak{D}_A(D)$, then $\| S_f \|_D^{(A)} \leq 6kA^2$. Conversely, if $\| S_f \|_D^{(A)} \leq 2kA^2$, then $f|_\Delta$ can be extended to a $k$-qc map of $\hat{C}$ for any $\Delta \in \mathfrak{D}_A(D)$.*

Combining this lemma with the results of preceding section, we obtain the following theorem (cf. [1]).

**Theorem 3.2.** *Let $D$ be a hyperbolic plane domain, $A \geq 1$ and $k \in [0, 1)$ be constants and $f : D \to \hat{C}$ be a meromorphic function.*

*If $f|_\Delta$ can be extended to a $k$-qc map of $\hat{C}$ for any $\Delta \in \mathfrak{D}_A(D)$ and if $D$ is simply connected, then $\| S_f \|_D \leq 96kA^2$. Conversely, if $\| S_f \|_D \leq 2kA^2$, then $f|_\Delta$ can be extended to a $k$-qc map of $\hat{C}$ for any $\Delta \in \mathfrak{D}_A(D)$.*

This result is crucial in author's paper [5] to estimate the norm of the Schwarzian derivative of a meromorphic map which is constructed by a certain qc-deformation and so has difficulties to calculate their derivatives.

**§4. The injectivity radius.** In this section we shall explain that the (local) norm of the Schwarzian derivative measures local injectivity. Let $d = d_D$ denote the hyperbolic distance on $D$ which is induced by the hyperbolic metric $\rho_D(z)\,|dz|$. It is well-known that, for the unit disk $U$, $d(0, z) = \dfrac{1}{2}\log\dfrac{1+|z|}{1-|z|} = \operatorname{arctanh}|z|$.

For each function $f$ which is meromorphic on a plane hyperbolic domain $D$ we let $\sigma(f) = \sigma_D(f)$ the injectivity radius of $f$, that is, $\sigma(f) = \dfrac{1}{2}\inf \{d_D(z_1, z_2)\,; f(z_1) = f(z_2),\ z_1 \neq z_2 \in D\}$. We remark that $\sigma(f) = \infty$ if $f$ is univalent. Firstly, in the case that $D = U$, Theorems 1.1 produces the next result due to Kra-Maskit [3], although the original form is apparently different from the next one.

**Proposition 4.1** (Kra-Maskit). *Let $f$ be a meromorphic function on the unit disk $U$ and $\sigma = \sigma(f) < \infty$. Then*

$$2\coth^2\sigma \leq \| S_f \|_U \leq 6\coth^2\sigma.$$

**Corollary 4.2** (cf. [6]). *For a meromorphic function $f$ on the unit disk $U$, $\| S_f \|_U < \infty$ if and only if $\sigma(f) > 0$.*

*Proof of Proposition* 4.1. By the hypothesis, for any $\sigma_1 > \sigma(f)$ there exist two points $z_1, z_2$ in $U$ such that $f(z_1) = f(z_2)$ and $0 < d(z_1, z_2) < 2\sigma_1$. Let $z_0$ be the midpoint of the hyperbolic segment joining $z_1$ and $z_2$ and $\Delta$ denote the hyperbolic disk $\{z\,; d(z_0, z) < \sigma_1\}$.

Since $f|_\Delta$ is not univalent, $\| S_f \|_\Delta > 2$ by Theorem 1.1. Let $A = \coth\sigma_1$ and choose $T \in$ Möb such that $T(U) = U$ and $T(0) = z_0$. It follows from the identity $S_{f\circ T} = S_f \circ T \cdot (T')^2$ that

$$\| S_f \|_U = \| S_{f\circ T} \|_U \geq \| S_{f\circ T} \|_U^{(A)} \geq A^2 \| S_{f\circ T} \|_{T^{-1}(\Delta)} = A^2 \| S_f \|_\Delta > 2A^2,$$

where we should note that $T^{-1}(\Delta) = B(0, \tanh\sigma_1) \in \mathfrak{D}_A(U)$. Because $\sigma_1 > \sigma(f)$ is arbitrary, we have the first inequality.

Next, we shall prove the second inequality. Let $z_0 \in U$ be any point and

$\Delta = \{z \in U \,;\, d(z_0, z) < \sigma\}$, where $\sigma = \sigma(f)$. By the hypothesis, $f|_\Delta$ is univalent, so we have $\| S_f \|_\Delta \le 6$ by the Nehari-Kraus theorem. We again choose $T \in$ Möb such that $T(U) = U$ and $T(0) = z_0$. Then we have

$$\begin{aligned} | S_f(z_0) | \rho_U(z_0)^{-2} &= | S_{f\circ T}(0) | \rho_U(0)^{-2} = | S_{f\circ T}(0) | \rho_{T^{-1}(\Delta)}(0)^{-2} \coth^2 \sigma \\ &\le \| S_{f\circ T} \|_{T^{-1}(\Delta)} \coth^2 \sigma = \| S_f \|_\Delta \coth^2 \sigma \le 6 \coth^2 \sigma, \end{aligned}$$

and proof is completed.

**Example.** Let $R$ be a hyperbolic Riemann surface and $\pi : U \to R$ be a holomorphic universal covering map of $R$. For simplicity, suppose that $R$ is of (topologically) finite type. Then, by Corollary 4.2,

$$\| S_\pi \|_U < \infty \Leftrightarrow \sigma(\pi) > 0 \Leftrightarrow R \text{ has no punctures.}$$

Generally, if $R$ has a puncture, then $\| S_\pi \|_U = \infty$ by the same reasoning.

Secondly, we return to the case of a general hyperbolic domain $D$.

**Proposition 4.3.** *Let $f$ be a meromorphic map on $D$ with positive injectivity radius $\sigma = \sigma_D(f)$. Then $\| S_f \|_D^{(\infty)} \le 6 \coth^2 \sigma$.*

*Proof.* We set $A = \coth \sigma$. Let $z_0$ be an arbitrary point in $D$ and $K$ be the Poincaré disk $\{z \in D \,;\, d_D(z, z_0) < \sigma\}$. Then, we note that $f$ is univalent in $K$. Set $r_0 = \inf_{z\in\partial K} | z - z_0 |$, $r_1 = \inf_{z\in\partial D} | z - z_0 |$ and $\Delta_j = B(z_0, r_j)$ $(j = 0, 1)$.

By the monotonicity, $d_D(z, z_0) \le d_{\Delta_1}(z, z_0)$, and hence

$$\sigma = \inf_{z\in\partial K} d_D(z, z_0) \le \inf_{z\in\partial K} d_{\Delta_1}(z, z_0) = \inf_{z\in\partial K} \operatorname{arctanh} \frac{| z - z_0 |}{r_1} = \operatorname{arctanh} \frac{r_0}{r_1}.$$

Thus we conclude that $\tanh \sigma \le \frac{r_0}{r_1}$, that is, $\frac{r_1}{r_0} \le A = \coth \sigma$. If we now let $\Delta = B(z_0, r) \in \mathfrak{D}_A(D)$, then $\frac{r_1}{r} \ge A \ge \frac{r_1}{r_0}$, so $r \le r_0$. This yields that $\Delta \subset \Delta_0 \subset K$, and hence $f|_\Delta$ is univalent. So, the Nehari-Kraus theorem implies that $\| S_f \|_\Delta \le 6$. Therefore $\| S_f \|_D^{(A)} \le 6A^2 = 6 \coth^2 \sigma$. The statement now readily follows from Theorem 2.1 (b).

**Remark.** Proposition 4.3 is, in some sense, a rough estimate because the injectivity radius $\sigma_D(f)$ is conformally invariant, while the norm of the Schwarzian derivative is *not*.

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