92. *On the Local Regularity of Solutions to the Simultaneous Relations Characterizing the Supporting Functions of Convex Curves of Constant Angle*

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Abstract: We shall define a curve of constant angle α, $0 < \alpha < \pi$ in the plane $\boldsymbol{R}^2$. This curve is a closed convex curve parametrized by $\theta \in \boldsymbol{T} = \boldsymbol{R}/2\pi\boldsymbol{Z}$ and characterized by a C^1 function $p(\theta)$ called the *supporting function*. We shall show that $\ddot{p}(\theta)$, the second derivative of $p(\theta)$ in the sense of distributions of L. Schwartz, belongs to L^∞. This result is the best possible one if the angle α is general.

Key words: local regularity; supporting function.

1. Characteristic function χ_α and modified characteristic function $\tilde{\chi}_\alpha$. Let α be a given angle $0 < \alpha < \pi$. Put $\hat{\alpha} = \pi - \alpha$. We use the notations

(1.1) $\quad c_1(\alpha) = \sin\alpha,\ c_2(\alpha) = \cos\alpha,\ \tilde{c}_1(\alpha) = \sin\alpha/2,\ \tilde{c}_2(\alpha) = \cos\alpha/2$

and we omit the variable as far as there is no confusion. Let $\Omega_\alpha = \min\{\tilde{c}_1, \tilde{c}_2\}$. The open intervals I_α and J_α are defined as follows:

$$I_\alpha = (-\Omega_\alpha, \Omega_\alpha), \tag{1.2}$$

$$J_\alpha = \begin{cases} (0, c_1) & \text{for } 0 < \alpha \leq \pi/2 \\ (-c_2, 1) & \text{for } \pi/2 \leq \alpha < \pi. \end{cases} \tag{1.3}$$

The *characteristic function* χ_α and the *modified characteristic function* $\tilde{\chi}_\alpha$ are defined by the formulas

$$\chi_\alpha(t) = c_1(1-t^2)^{1/2} - c_2 t,\ t \in J_\alpha; \tag{1.4}$$

$$\tilde{\chi}_\alpha(s) = \tilde{c}_1(1-s^2)^{1/2} - \tilde{c}_2 s,\ s \in I_\alpha \quad \text{or} \quad s \in J_\alpha. \tag{1.5}$$

We state some properties of these functions without proofs.

Proposition 1.1. *χ_α maps J_α onto J_α and is strictly monotone decreasing. χ_α has the only one fixed point $\tilde{c}_1$. Its inverse mapping $\overset{-1}{\chi_\alpha}$ coincides with χ_α. $\tilde{\chi}_\alpha$ maps J_α onto I_α and is strictly monotone decreasing. $\tilde{\chi}_\alpha$ maps $\tilde{c}_1$ to 0. Its inverse mapping $\overset{-1}{\tilde{\chi}_\alpha}$ has the same expression as $\tilde{\chi}_\alpha$.*

$\tilde{\chi}_\alpha$ has the linearization effect on χ_α as follows:

Proposition 1.2. *If w belongs to I_α, p belongs to J_α, and $w = \tilde{\chi}_\alpha(p)$, then $\tilde{\chi}_\alpha(\chi_\alpha(p)) = -w$.*

2. Curves of constant angle α. Let C be the circle of radius r with the center at the origin of the plane $\boldsymbol{R}^2$, and call it the *director circle*. (This terminology comes from the classical example of ellipses, that is, $\alpha = \pi/2$.) Hereafter we assume $r = 1$, without loss of generality. Let A be a figure contained in C. A figure simply means here a subset of $\boldsymbol{R}^2$. For a point P on C, we put

$$C(P; A) = \{\text{ray; starting from } P, \text{ passing through a point of } A\},$$

where a ray means a closed half line. $C(P;A)$ is called the *sight-cone* at P for A. We assume that $C(P;A)$ is a closed convex cone with angle α at the vertex P. Suppose that the angle α at P is independent of P. Then, there exists a closed convex set D with non-empty interior such that $\partial D \subseteq A \subseteq D$, where ∂D designates the boundary of D. (In fact, $D = \cap_{p\in C} C(P;A)$ and the origin O lies in the interior of D.) $\Lambda = \partial D$ is a closed convex curve by definition. It is clear that $C(P;\Lambda) = C(P;A) = C(P;D)$ for every P on C. Thus, if we neglect the internal structure of A, it is enough to study D or $\Lambda = \partial D$. Λ is in fact a *strictly convex* curve, that is, no part of it is a straight line segment. We call Λ a *convex curve of constant angle* α with the director circle C.

In general, to characterize a closed convex curve in $\boldsymbol{R}^2$, it is enough to obtain its supporting function $p(\theta)$ defined by

$$p(\theta) = \sup_{(x,y)\in\Lambda} (x\cos\theta + y\sin\theta). \tag{2.1}$$

It is well known that if Λ is strictly convex, then p is C^1-function with period 2π. Λ has the following *parametric representation*:

$$\begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} p(\theta) \\ \dot{p}(\theta) \end{pmatrix}, \ 0 \le \theta \le 2\pi, \tag{2.2}$$

where we denote $\dot{p}(\theta) = dp(\theta)/d\theta$. This is a continuous closed curve, that is, $(x(\theta), y(\theta))$ depends continuously on θ but not C^1 in θ in general. In fact, in the present convex case, the second derivative $\ddot{p}(\theta)$ in the sense of distributions of L. Schwartz satisfies the inequality: $p + \ddot{p} \ge 0$, where $p + \ddot{p}$ is, if p is C^2, the radius of curvature of Λ. This means that the left-hand side is a non-negative Radon measure. We can characterize convex curves of constant angle α by $p(\theta)$ as follows. But we omit the proof.

Theorem 2.1. *A continuous function $p(\theta)$ of θ is the supporting function of a convex curve of constant angle α if and only if $p(\theta)$ satisfies the following four conditions*:

(1°) *$p(\theta)$ is a function with period 2π.* [*periodicity*]
(2°) *For every θ, $p(\theta)$ belongs to J_α.* [*inequality*]
(3°) $p(\theta + \pi - \alpha) = \chi_\alpha(p(\theta))$. [*functional equation*]
(4°) $p + \ddot{p} \ge 0$. [*differential inequality*]

Remark. The last differential inequality is the inequality in the sense of distributions of L. Schwartz. Thus, if we put $\mu = p + \ddot{p}$ then we get $\mu \ge 0$, that is, μ is a non-negative Radon measure. We can replace these two conditions for (4°). If Λ is unknown, then both p and μ are unknown. Hence we have a system of five relations for two unknown quantities.

For every α, $0 < \alpha < \pi$, there exists a convex curve of constant angle α. In fact, if we employ the function $p(\theta) \equiv \tilde{c}_1$, we get a circle of radius $\tilde{c}_1$ concentric with the director circle C. We call this the *trivial* curve of constant angle α.

The totality of functions p satisfying the four conditions of the above theorem is denoted by $\mathscr{P}_\alpha$ or more precisely $\mathscr{P}_\alpha^{\text{convex}}$. When $\mathscr{P}_\alpha$ is a singleton $\mathscr{P}_\alpha = \{\tilde{c}_1\}$ is not interesting. The following theorem, whose proof shall be published elsewhere, is the answer to the question: "Is $\mathscr{P}_\alpha = \{\tilde{c}_1\}$?"

Theorem 2.2. (I) *If α/π is irrational, then $\mathscr{P}_\alpha = \{\tilde{c}_1\}$.*
(II) *Suppose that α/π is rational and m/n is its irreducible fraction representation.*
(i) *If mn is odd, then $\mathscr{P}_\alpha = \{\tilde{c}_1\}$.*
(ii) *If mn is even, then $\{\tilde{c}_1\}$ is a proper subset of $\mathscr{P}_\alpha$.*

3. Local regularity. Now we state and prove the main theorem.

Theorem 3.1. *The supporting function $p(\theta)$ of a convex curve of constant angle α belongs to C^1 and its second derivative $\ddot{p}$ belongs to L^∞. This result is the best possible one if the angle α is general.*

Proof. By a routine work in the elementary geometry, we can show that if a convex curve Λ is of constant angle α, then it is strictly convex. Hence its supporting function p belongs to C^1.

Next we shall show that $\ddot{p}$ belongs to L^∞. We use the same notation as in Theorem 2.2. By Theorem 2.2, we can reduce to the case that mn is even, because the other cases are trivial. Replace θ by $\theta + \pi - \alpha$ in (3°) in Theorem 2.1 and use (3°), then Proposition 1.1 implies that

$$\begin{aligned} p(\theta + 2(\pi - \alpha)) &= \chi_\alpha(p(\theta + \pi - \alpha)) \\ &= \chi_\alpha(\chi_\alpha(p(\theta))) = p(\theta). \end{aligned}$$

Since $p(\theta)$ is 2π-periodic, we have for every integer k,

$$\begin{aligned} p(\theta) &= p((\theta + 2\alpha) - 2\alpha) \\ &= p(\theta + 2\alpha) = p(\theta + 2k\alpha), \end{aligned}$$

that is, 2α is a period of $p(\theta)$. Hence for all integers k, l

$$\begin{aligned} p(\theta) &= p(\theta + 2l\pi + 2k\alpha) \\ &= p(\theta + 2l\pi + 2k(m/n)\pi) \\ &= p(\theta + (ln + km)2\pi/n). \end{aligned}$$

Choose k, l so that $ln + km = 1$, then $p(\theta + 2\pi/n) = p(\theta)$, that is, $2\pi/n$ is a period of $p(\theta)$. Since $\pi - \alpha = (n - m)\pi/n$ and $n - m$ is odd, (3°) in Theorem 2.1 implies that

(3.1) $$p(\theta + \pi/n) = \chi_\alpha(p(\theta)).$$

Put $q(\theta) = p(\theta + \pi/n)$, $\mu = p + \ddot{p}$, and $\nu = q + \ddot{q}$. Then (4°) in Theorem 2.1 implies that $\nu \geq 0$. Since we can justify the following Leibniz formula: $(fg)^{\cdot} = \dot{f}g + f\dot{g}$, where f is continuous and g is of bounded variation, that is, $\dot{g}$ is a Radon measure, if we differentiate (3.1) twice then we have

(3.2) $$\nu = \chi_\alpha(p) + \chi''_\alpha(p)\,(\dot{p})^2 + \chi'_\alpha(p)\ddot{p}.$$

Hence

(3.3) $$\nu - \chi'_\alpha(p)\mu = \chi_\alpha(p) + \chi''_\alpha(p)\,(\dot{p})^2 + \chi'_\alpha(p)p.$$

Consider the Lebesgue decomposition of both sides of (3.3) with respect to the Lebesgue measure. Since the right-hand side of (3.3) is absolutely continuous, it implies that

(3.4) $$\operatorname{sing}(\nu - \chi'_\alpha(p)\mu) = 0,$$

where sing denotes the singular part of a Radon measure. On the other hand, $\chi'_\alpha(p) < 0$ implies that

(3.5) $$\operatorname{sing}\nu = \operatorname{sing}\mu = 0,$$

because singular parts of non-negative Radon measures are also non-negative. Hence both ν and μ are locally summable. The left-hand side of (3.3) is bounded below, because it is non-negative. The right-hand side of (3.3) is bounded above, because it is continuous and periodic. Thus ν and μ are essentially bounded and therefore $\ddot{p}$ is also essentially bounded.

Finally we shall construct a supporting function p whose second derivative $\ddot{p}$ is *essentially everywhere discontinuous*, that is, however the values of $\ddot{p}$ on any sets of measure zero are altered, $\ddot{p}$ remains everywhere discontinuous. This example is enough to show that the local regularity of $\ddot{p}$ is the best possible one in general. Put

$$w(\theta) = \tilde{\chi}_\alpha(p(\theta)), \tag{3.6}$$

where $\tilde{\chi}_\alpha$ is defined by (1.5). Then (3.1) implies that

$$w(\theta + \pi/n) = -w(\theta) \tag{3.7}$$

and Proposition 1.1 implies that

$$p(\theta) = \tilde{\chi}_\alpha(w(\theta)) \tag{3.8}$$

and the second derivative of (3.8) is

$$\ddot{p}(\theta) = \tilde{\chi}''_\alpha(w(\theta))(\dot{w}(\theta))^2 + \tilde{\chi}'_\alpha(w(\theta))\ddot{w}(\theta). \tag{3.9}$$

Since $\tilde{\chi}_\alpha$ is real analytic, (3.9) implies that $\ddot{p}(\theta)$ is discontinuous at θ_0 if $\dot{w}(\theta)$ is continuous at θ_0 and $\ddot{w}(\theta)$ is discontinuous at θ_0. Therefore if we construct a function $w(\theta)$ satisfying (3.7) and $w(\theta) \in I_\alpha$ such that $\dot{w}$ is continuous and $\ddot{w}$ is essentially everywhere discontinuous, then we have a supporting function $p(\theta)$ whose second derivative $\ddot{p}$ is essentially everywhere discontinuous. Let us construct the function $w(\theta)$ having the above property. First we construct $v_j(\theta)$, $j = 0,1,2$, satisfying $v_0(\theta) \in I_\alpha$ and

$$v_j(\theta + \pi/n) = -v_j(\theta),\ j = 0,1,2. \tag{3.10}$$

For arbitrary L^∞ function $u(\theta)$ on $[0, \pi/n]$, we may assume that $u(0) = 0$ and $u(\pi/n) = 0$ without loss of generality, we extend $u(\theta)$ first to an odd function on $[-\pi/n, \pi/n]$ and next to a $2\pi/n$-periodic function on $[0, 2\pi]$. Then $u(\theta + \pi/n) = -u(\theta)$. Put $v_0(\theta) = \varepsilon u(\theta)$ for $\varepsilon > 0$ and choose ε so small that $v_0(\theta) \in I_\alpha$. Put

$$v_1(\theta) = \int_0^\theta v_0(\tau)d\tau - \frac{1}{2}\int_0^{\pi/n} v_0(\tau)d\tau$$

and

$$v_2(\theta) = \int_0^\theta v_1(\tau)d\tau - \frac{1}{2}\int_0^{\pi/n} v_1(\tau)d\tau.$$

Then (3.10) are satisfied. Hence if we put $w(\theta) = \eta v_2(\theta)$ for $\eta > 0$, then (3.7) is satisfied. Choose η so small that $w(\theta) \in I_\alpha$. Since we can construct an essentially everywhere discontinuous function $u(\theta)$ on $[0, \pi/n]$, whose example shall be given in Appendix, it implies that $w(\theta)$ has the desired property. Q.E.D.

Appendix. In this appendix, we shall give an example of an essentially everywhere discontinuous L^∞ function on $\boldsymbol{R}$. Put

$$h_0(x) = \begin{cases} -\frac{1}{2}\log|x|, & -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}.$$

Then $h_0(x) \geq 0$ and $\int_{-\infty}^{\infty} h_0(x)\,dx = 1$. Fix a numbering of the rational numbers $\boldsymbol{Q} = \{r_n\,;\, n = 1,2,3,\ldots\}$ and put

$$h(x) = \sum_{n=1}^{\infty} h_0(x - r_n)/2^n.$$

Then $h(x)$ is unbounded on every non-void open interval. Since

$$\int_{-\infty}^{\infty} h(x)\,dx = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} h_0(x - r_n)\,dx/2^n = 1,$$

it implies that $h(x)$ is summable. However the values of $h(x)$ on any sets of measure zero are altered, $h(x)$ remains discontinuous at every point in $\boldsymbol{R}$. Hence if we put

$$g(x) = \frac{2}{\pi} \tan^{-1} h(x),$$

then $g(x)$ has the desired property.

Reference

[1] Matsuura, S.: On non-convex curves of constant angle. Functional Analysis and Related Topics, 1991. Lect. Notes in Math., vol. 1540, Springer-Verlag, pp. 251–268 (1993).