A Table of Absolute Norms of Heilbronn Sums

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Let p be an odd prime number and ζ a primitive p^2th root of unity. Let L be the unique subfield of $\boldsymbol{Q}(\zeta)$ of degree p. The pth Heilbronn sum is defined as the trace of ζ from $\boldsymbol{Q}(\zeta)$ to L. We denote by NH_p its absolute norm. Fouché[1] proved that if l is a prime divisor of NH_p, then l satisfies the congruence

$$l^{p-1} \equiv 1 \pmod{p^2}.$$

This congruence is well known for $l = 2$, because Wieferich [4] proved that if there exists a counterexample to the first case of Fermat's last theorem for the exponent p (FLT,I,p), then $l = 2$ satisfies the congruence above. Wieferich's result has been generalized as follows [3]: if there exists a counterexample to (FLT,I,p), then all prime numbers l with $2 \leqq l \leqq 113$ satisfy the congruence above. In [1], the table of the values of NH_p for $p < 50$ was given (by the referee). We extend it to $p < 100$ and give complete factorizations of NH_p. The computation was done on a NeXT computer.

Table

p	NH_p (factorization)
3	-1
5	-1
7	97 (prime)
11	$-243 = -3^5$
13	$12167 = 23^3$
17	577 (prime)
19	221874931 (prime)
23	157112485811 (prime)
29	$-2480435158303 = -137 \cdot 18105366119$
31	$310695313260929 = 3727 \cdot 34367 \cdot 2425681$
37	$-5114055181947668782 9$ (prime)
41	$2727257042363914863401 = 17435281 \cdot 156421742922521$
43	$-2572343484535669027372727 = -19^4 \cdot 19738518615846018887$
47	$1052824394331287344099620777449 = 53^2 \cdot 374803985165997630508942961$

Continued

p	NH_p(factorization)
53	24872084253252533462383654866641837
	$= 15692059 \cdot 1585010880551273319988387 43$
59	$-$ 38591939342288398936143323247422 8039
	$= -53^2 \cdot 191559377 \cdot 717201928003392149777423$
61	28895861850580295495180392084567837722 67
	$= 2281 \cdot 5749 \cdot 4431029 \cdot 9582073 \cdot 94339087 \cdot 55012619117$
67	$-$ 33325854862163830411598110208119480777049877 91
	(prime)
71	$-$ 15940589232607031072946301497035295916917925 9
	$= -11^4 \cdot 161323 \cdot 498577 \cdot 3607523 \cdot 83966159 \cdot 446880809176117$
73	$-$ 53638311006466780577262790821628723384059340479403
	$= -523605697 \cdot 102440273881257599413137765805532714408299$
79	1040611956630950417455971245710339238106740379250667
	$= 31^5 \cdot 5872927 \cdot 6189074918318922117260814024643164971$
83	$-$ 363791338869557051008173120274327051451525936368808 27
	$= -821 \cdot 77069 \cdot 2611457 \cdot 37082337983851 \cdot 593717003744478522862828 9$
89	785275890160998487533839631209769637863435220503329353960405381 73
	$= 86955810683941 \cdot 903074658248253496993647901544985025793732684727353$
97	6987145228295591002299725696981364375137003233828984685582960524329359233
	$= 107^4 \cdot 5330459640547418970477126869635012373234949985718208499568869243 3$

Remark. In [2] Ihara tentatively defines the differential $d\alpha$ of nonzero number α which is not a root of unity in an algebraic number field k as a function on the set of the finite primes of k with value in $(\mathfrak{p}/\mathfrak{p}^2) \cup \{\infty\}$ for each prime $\mathfrak{p}$. According to his definition, the congruence

$$\alpha^{N(\mathfrak{p})-1} \equiv 1 \pmod{\mathfrak{p}^2}$$

is equivalent to $d\alpha(\mathfrak{p}) = 0$ for α with $\mathrm{ord}_{\mathfrak{p}}\alpha = 0$, where $N(\mathfrak{p})$ denotes the absolute norm of $\mathfrak{p}$ and $\mathrm{ord}_{\mathfrak{p}}$ the normalized additive $\mathfrak{p}$-adic valuation. Thus, when l is a prime divisor of NH_p, p can be considered as a zero of the differential dl, and therefore we have got some big prime numbers l whose differential dl has a small zero. We do not know when NH_p has a big prime divisor, however, any other method of getting big prime numbers whose differential has a given small zero seems to be unknown.

References

[1] W. L. Fouché: Arithmetic properties of Heilbronn sums. J. Number Theory, **19**, 1–6 (1984).

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[3] J. Suzuki: On the generalized Wieferich criteria. Proc. Japan Acad., **70 A**, 230–234 (1994).

[4] A. Wieferich: Zum letzten Fermat'schen Theorem. J. reine angew. Math., **136**, 293–302 (1909).