Stable Limit Distributions over a Nilpotent Lie Group

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In the previous paper [3], the author defined a convolution semigroup $\{\mu_t\}_{t>0}$ of stable distributions over a simply connected nilpotent Lie group G in connection with a dilation $\{\gamma_r\}_{r>0}$. It corresponds to a convolution semigroup of strictly operator-stable distributions in the case where G is a Euclidean space. In this paper, motivated by Sharpe [4], we shall characterize stable distributions over a Lie group as a certain limit distribution. We show that our definition of stable distributions coincides with that given in [3], provided that the distributions are full. Then we shall discuss the domain of the normal attraction of stable distributions over a simply connected nilpotent Lie group.

1. Stable distributions and associated convolution semigroup. Let G be a Lie group and let $\mathcal{G}$ be its left invariant Lie algebra. For two (probability) distributions μ and ν over G, their *convolution* is defined by

$$\mu * \nu(E) = \int_G \nu(\sigma^{-1}E)\mu(d\sigma).$$

The n-times convolution of μ is denoted by μ^{n*}. Let φ be a continuous map from G (or $\mathcal{G}$) into G (or $\mathcal{G}$). For a distribution μ over G (or $\mathcal{G}$), we define a distribution $\varphi\mu$ by $\varphi\mu(E) = \mu(\varphi^{-1}(E))$. Let β be an automorphism of G, i.e., $\beta : G \to G$ is a diffeomorphism and satisfies $\beta(\sigma\tau) = \beta(\sigma)\beta(\tau)$ for any $\sigma, \tau \in G$. Then we have the relation $\beta(\mu * \nu) = \beta\mu * \beta\nu$ for any distributions μ and ν over G. A distribution μ over G (or $\mathcal{G}$) is called *full* if μ is not supported by any proper subgroup of G (or proper subalgebra of $\mathcal{G}$).

Let $N = \{1,2,\dots\}$ be the set of all positive integers. Let $\{\beta_n\}_{n\in N}$ be a sequence of automorphisms of G. It is called a *semigroup* if $\beta_k\beta_l = \beta_{kl}$ holds for all $k, l \in N$. A distribution μ over G is called *stable* if there exists a sequence $\{\beta_n\}_{n\in N}$ of automorphisms of G and a distribution ν over G such that $\beta_n\nu^{n*}$ converges weakly to μ as $n \to \infty$.

We will give a characterization of stable distributions in the case where the Lie group is simply connected and nilpotent. It is known that if G is a simply connected nilpotent Lie group, the exponential map: $\exp: \mathcal{G} \to G$ is a diffeomorphism. Hence G is non-compact. Denote the inverse map of exp by log. Then μ over G is full if and only if $\log \mu$ over $\mathcal{G}$ is full.

Theorem 1.1. *Let μ be a full distribution over a simply connected nilpotent Lie group G. Then μ is stable if and only if there exists a sequence $\{\gamma_k\}_{k\in N}$ of automorphisms of G such that $\mu^{k*} = \gamma_k\mu$ holds for all $k \in N$*

Before we proceed to the proof of the theorem, we need two facts. Let β' be a linear map of $\mathcal{G}$. It is called an *automorphism of* $\mathcal{G}$ if it is a one to one, onto map and satisfies $\beta'[X, Y] = [\beta'X, \beta'Y]$ for all $X, Y \in \mathcal{G}$. Now if β is an automorphism of G, the differential $d\beta$ defines an automorphism of $\mathcal{G}$. Conversely let β' be an automorphism of $\mathcal{G}$. If G is simply connected and nilpotent, there exists a unique automorphism of G such that its differential coinsides with β'. Indeed, define $\beta : G \to G$ by $\beta(\exp X) = \exp \beta'X$. Then, using Champbel-Hausdorff formula we have

$$\begin{aligned}&\beta(\exp X \exp Y)\\ &= \beta(\exp(X + Y + 1/2[X, Y] + \cdots))\\ &= \exp(\beta'X + \beta'Y + 1/2[\beta'X, \beta'Y] + \cdots)\\ &= \exp(\beta'X)\exp(\beta'Y) = \beta(\exp X)\beta(\exp Y).\end{aligned}$$

Therefore β is an automorphism of G and satisfies $d\beta = \beta'$. The uniqueness will be obvious.

Another fact we need is stated in the following lemma.

Lemma (cf. Sharpe [4] and Eurek-Mason [1]). *Let $\{m^{(n)}\}$ be a sequence of distributions over the Lie algebra $\mathcal{G}$ converging weakly to a full distribution m. Suppose that there exists a sequence $\{\beta^{(n)}\}$ of automorphisms of $\mathcal{G}$ such that $\beta^{(n)}m^{(n)}$ converges weakly to a full distribution $\tilde{m}$. Then a certain subsequence $\{\beta^{(n')}\}$ converges to an automorphism β of $\mathcal{G}$ such that $\beta m = \tilde{m}$.*

Proof. Let V be the linear support of m, spanned by $\{Y_1,\dots,Y_r\}$, which generates the Lie algebra $\mathcal{G}$. We can show similarly as in [1] Lemma 2.2.2, that the sequence $\{\beta^{(n)}Y_i\}$ is bounded for any $i=1,\dots,r$. Since any element X_j of the basis $\{X_1,\dots,X_d\}$ of $\mathcal{G}$ is written as a linear sum of the elements of the forms

$$[Y_{i_1},[Y_{i_2},[\cdots,[Y_{i_{m-1}},[Y_{i_m}]]\cdots]$$

$(i_1,\dots,i_m\in\{1,\dots,r\})$, the sequence $\{\beta^{(n)}X_j\}$ is also bounded for any j. Consequently, a subsequence $\{\beta^{(n')}\}$ of $\{\beta^{(n)}\}$ converges to an endomorphism β of $\mathcal{G}$. Then the sequence $\{\beta^{(n')}m^{(n')}\}$ converges weakly to βm and satisfies $\beta m=\tilde{m}$. Since βm is full, β is an automorphism.

Proof of Theorem 1.1. "If" part is obvious. We shall prove the "only if" part. Suppose that μ is stable. Then there exists a sequence $\{\beta_n\}_{n\in N}$ of automorphisms of G and a distribution ν over G such that $\mu=\lim_{n\to\infty}\beta_n\nu^{n*}$. Then we have $\mu^{k*}=\lim_{n\to\infty}(\beta_n\nu^{n*})^{k*}$ for any positive integer k. Note that $(\beta_n\nu^{n*})^{k*}=\beta_n\nu^{nk*}$. Then we obtain $\mu^{k*}=\lim_{n\to\infty}(\beta_n\beta_{nk}^{-1})\beta_{nk}\nu^{nk*}$. For each $k\in N$, set $\eta_k^{(n)}=\beta_{nk}\nu^{nk*}$ and $\gamma_k^{(n)}=\beta_n\beta_{nk}^{-1}$. Then we have $\eta_k^{(n)}\to\mu$ and $\gamma_k^{(n)}\eta_k^{(n)}\to\mu^{k*}$ as $n\to\infty$.

Let $d\gamma_k^{(n)}$ be the differential of $\gamma_k^{(n)}$. Then $\gamma_k^{(n)}(\exp X)=\exp(d\gamma_k^{(n)}X)$, or equivalently, $\log\gamma_k^{(n)}(\sigma)=d\gamma_k^{(n)}\log\sigma$. Therefore we have $\log\gamma_k^{(n)}\eta_k^{(n)}=d\gamma_k^{(n)}\log\eta_k^{(n)}$. Define distributions over the Lie algebra $\mathcal{G}$ by $m^{(n)}=\log\eta_k^{(n)}$, $m=\log\mu$ and $\tilde{m}=\log\mu^{k*}$. Then we have $m^{(n)}\to m$ and $d\gamma_k^{(n)}m^{(n)}\to\tilde{m}$ as $n\to\infty$. Since m is full, $\tilde{m}$ is also full. Therefore, there exists an automorphism γ'_k on $\mathcal{G}$ such that $\gamma'_k m=\tilde{m}$ by the above lemma. We can choose $\{\gamma'_k\}_{k\in N}$ such that $\gamma'_k\gamma'_l=\gamma'_{kl}$ for all $k,l\in N$. Now for each $k\in N$ define an automorphism γ_k of G by $\gamma_k(\sigma)=\exp(\gamma'_k\log\sigma)$. Then $\{\gamma_k\}_{k\in N}$ is a semigroup of automorphisms satisfying $\gamma_k\mu=\mu^{k*}$ for all $k\in N$. The proof is complete.

Let $\{\mu_t\}_{t>0}$ be a family of distributions over G. It is called a *convolution semigroup* if it satisfies (a) $\mu_t*\mu_s=\mu_{t+s}$ for any $s,t>0$, and (b) $\mu_t\to\delta_e$ as $t\to 0$, where δ_e is a unit measure at the point e (identity of G). In particular if each μ_t is a stable distribution, it is called a *convolution semigroup of stable distributions.*

Let $\{\gamma_t\}_{t>0}$ be a family of automorphisms of G. It is called a *one parameter group* if $\gamma_t(\sigma)$ is continuous in $(0,\infty)\times G$ and satisfies $\gamma_t\gamma_s=\gamma_{ts}$ for all $t,s>0$. Further, if $\gamma_t(\sigma)\to e$ holds uniformly on compact sets of G as $t\to 0$, it is called a *dilation.* It is known that if a dilation exists on a Lie group G, it is simply connected and nilpotent. See [3]. Given a dilation, the family of differentials $\{d\gamma_t\}_{t>0}$ defines a one parameter group of automorphisms of $\mathcal{G}$. Further, there exists a linear map $Q:\mathcal{G}\to\mathcal{G}$ such that $d\gamma_t=\exp(\log t)Q\equiv t^Q$. The linear map Q is called the *exponent* of the dilation. Note that real parts of eigen values of Q are all positive.

Theorem 1.2. *Let μ be a full stable distribution over a simply connected nilpotent Lie group G. Then there exists a unique convolution semigroup $\{\mu_t\}_{t>0}$ of stable distributions such that $\mu_1=\mu$. Furthermore there exists a dilation $\{\gamma_t\}_{t>0}$ such that $\mu_t=\gamma_t\mu$ holds for all $t>0$.*

Proof. Let $\{\gamma_k\}_{k\in N}$ be the sequence of automorphisms defined in Theorem 1.1. We first consider the case where it is a semigroup. For $k,l\in N$, we set $\gamma_{l/k}=\gamma_k^{-1}\gamma_l$. It is well defined since $\gamma_{mk}^{-1}\gamma_{ml}=\gamma_k^{-1}\gamma_l$ holds for all $m\in N$. Then $\{\gamma_r\}_{r\in Q^+}$ (positive rationals) is a one parameter group of automorphisms of G. Let $t>0$ be an arbitrary real number. Then there exists a sequence of positive rationals $\{r_n\}$ such that $\{\gamma_{r_n}\}$ converges to an automorphism γ_t. We can prove that γ_t does not depend on the choice of sequences $\{r_n\}$ converging to t, and $\{\gamma_t\}_{t>0}$ satisfies $\gamma_s\gamma_t=\gamma_{st}$ for all $s,t>0$. Moreover, γ_t is continuous in t.

Now for each $t>0$, define a distribution μ_t by $\mu_t=\gamma_t\mu$. Then $\{\mu_t\}_{t>0}$ satisfies $\mu_t*\mu_s=\mu_{s+t}$ for all $s,t>0$. Indeed, if s,t are rationals such that $s=k/n$ and $t=l/n$, we have $\mu_{k/n}*\mu_{l/n}=\gamma_n^{-1}\mu^{k*}*\gamma_n^{-1}\mu^{l*}=\gamma_n^{-1}\mu^{(k+l)*}=\mu_{(k+l)/n}$. Therefore $\{\mu_t\}_{t>0}$ satisfies $\mu_s*\mu_t=\mu_{s+t}$ for positive rationals s,t. Since μ_t is continuous in $t>0$, the equality holds for all positive reals s,t. Therefore $\{\mu_t\}_{t>0}$ has the convolution property. We have further, $\mu_t=\gamma_{t/n}\mu^{n*}$, so that μ_t is stable for all $t>0$. We shall prove $\mu_t\to\delta_e$ as $t\to 0$. Let $\bar{G}=G\cup\{\infty\}$ be the one point compactification of G. Then it is a topological semigroup by setting $\sigma\infty=\infty\sigma=\infty$ and $\infty\infty=\infty$. For each $t>0$, μ_t can be considered as a measure on the compact space $\bar{G}$. Now let μ_0 be any accumulation point of $\{\mu_t\}_{t>0}$ as $t\to 0$. It is a distribution over $\bar{G}$ and satisfies $\mu_0=\mu_0*\mu_0$, which implies $\mu_0=\delta_e$ or $\mu_0=\delta_\infty$. We have further $\mu_t*\mu_0=\mu_t$, which excludes the case $\mu_0=\delta_\infty$. This proves $\mu_t\to\delta_e$ as

$t \to 0$. Now since $\mu_t \to \delta_e$, we have $\log \mu_t \to \delta_0$. Note the equality $\log \mu_t = t^Q \log \mu$. Since $\log \mu$ is a full distribution, $t^Q \to 0$ as $t \to 0$ or equivalently $\gamma_t(\sigma) \to e$ uniformly on compact sets of G as $t \to 0$. Therefore $\{\gamma_t\}_{t>0}$ is a dilation.

Now in case where $\{\gamma_k\}_{k \in N}$ is not a semigroup, let $\mathscr{A}$ (or $\mathscr{N}$) be the group generated by automorphisms β of G such that $\beta\mu = \mu^{k*}$ for some $k \in N$ (or $\beta\mu = \mu$). Then $\mathscr{N}$ is a normal subgroup of $\mathscr{A}$. Consider the factor group $\mathscr{A}/\mathscr{N}$. Then $\tilde{\gamma}_k \equiv \gamma_k \mathscr{N}$, $k \in N$ define a semigroup in the factor group. A certain modification of the above argument shows that there exists a dilation $\{\gamma_t\}_{t>0}$ and $\mu_t \equiv \gamma_t \mu$, $t > 0$ defines a convolution semigroup of stable distributions.

Conversely suppose that we are given a convolution semigroup of stable distributions $\{\hat{\mu}_t\}_{t>0}$ such that $\hat{\mu}_1 = \mu$. Let $\{\gamma_t\}_{t>0}$ be the dilation constructed above. We will prove that $\hat{\mu}_t = \gamma_t \mu$ holds for all $t > 0$, which implies the uniqueness of the convolution semigroup $\{\hat{\mu}_t\}_{t>0}$. For each positive integer n, there exists an automorphism $\delta^{(n)}$ such that $\delta^{(n)}\hat{\mu}_{1/n} = \hat{\mu}_1 = \mu$ since $\hat{\mu}_{1/n}$ is stable. Then we have $\delta^{(n)}\mu = \hat{\mu}_n = \mu^{n*} = \gamma_n \mu$. Therefore $\gamma_n^{-1}\delta^{(n)} \in \mathscr{N}$. Since $\mathscr{N}$ is a normal subgroup of $\mathscr{A}$, there exists $\beta^{(n)} \in \mathscr{N}$ such that $\delta^{(n)} = \beta^{(n)}\gamma_n$. This equality implies $(\delta^{(n)})^{-1} = \gamma_{1/n}(\beta^{(n)})^{-1}$. Consequently, $\hat{\mu}_{1/n} = (\delta^{(n)})^{-1}\mu = \gamma_{1/n}\mu$, which implies $\hat{\mu}_{k/n} = \gamma_{k/n}\mu$ for any positive integer k. Since $\hat{\mu}_t$ is continuous in $t > 0$, we get the equality $\hat{\mu}_t = \gamma_t \mu$ for all real $t > 0$. The proof is complete.

2. Domain of normal attraction of stable distributions. Let G be a simply connected nilpotent Lie group equipped with a dilation $\{\gamma_t\}_{t>0}$. Let $\mathscr{G}$ be its Lie algebra, where an inner product $\langle\, , \rangle$ and the associated norm $|\ |$ are defined on $\mathscr{G}$. Let $\xi_1, \ldots, \xi_n, \ldots$ be a sequence of independent random variables with values in G with the identical distribution. Then the products $\psi_n = \xi_1 \cdots \xi_n$, $n = 1,2,\ldots$ define a random walk on the group G. We shall discuss the weak convergence of the sequence $\{\psi_n\}$ as $n \to \infty$ by constricting its spacial scale through the inverse of $\{\gamma_n\}$. Namely we consider a sequence of G-valued random variables $\varphi^{(n)} = \gamma_n^{-1}(\psi_n)$. Let $\mu^{(n)}$ be their distributions. If the sequence $\{\mu^{(n)}\}$ converges weakly, the limit distribution μ should be stable with respect to the dilation $\{\gamma_k\}$ in view of Theorem 1.1. The identical distribution ν of the random variables ξ_k is said to belong to the *domain of normal attraction of the stable distribution* μ. We are interested in finding criteria which makes ν to belong to the domain of normal attraction of a stable distribution.

For the study of the above problem, it is more convenient to consider a sequence of G-valued stochastic processes $\varphi_t^{(n)} = \gamma_n^{-1}(\psi_{[nt]})$ with continuous time parameter $t \in [0, \infty)$, instead of the sequence of G-valued random variables $\varphi^{(n)}$. If the sequence of the distributions of random variables $\varphi_t^{(n)}$ converges weakly for any $t > 0$, we say that the distributions of $\varphi_t^{(n)}$ converge weakly.

In order to introduce an assumption for the distribution π of $\eta_k \equiv \log \xi_k$, we need a fact on the exponent Q of the dilation. Let g be the minimal polynomial of Q. It is factorized as $g = g_1^{l_1} \cdots g_p^{l_p}$, where $g_1, \ldots, g_p$ are distinct irreducible monic polynomials and l_j are positive integers. Set $W_j = \mathrm{Ker}(g_j(Q)^{l_j})$, $j = 1, \ldots, p$. These are Q-invariant subspaces of $\mathscr{G}$ and admits a direct sum decomposition $\mathscr{G} = \sum_j \oplus W_j$. Let $\kappa_j = \alpha_j \pm \sqrt{-1}\beta_j$ (α_j, β_j are reals) be the roots of g_j (= eigen values of Q). We set

$I = \{j\,; \alpha_j = 1/2\}$, $J = \{j\,; 1/2 < \alpha_j < \infty\}$,
$J_1 = \{j\,; 1/2 < \alpha_j < 1\}$.

The subspaces of $\mathscr{G}$ are defined by $W_I = \oplus_{j \in I} W_j$ etc. and projectors to W_I, W_J etc. are denoted by T_{W_I}, T_{W_J} etc. We define
$S = \{\theta \in \mathscr{G} : |\theta| = 1, |r^Q\theta| > 1 \text{ for all } r > 1\}$.
Then every $X \in \mathscr{G}$ ($X \neq 0$) is represented uniquely by $X = r^Q\theta$, where $\theta \in S$ and $r \in (0, \infty)$. We denote r and θ by $r(X)$ and $\theta(X)$. We set $S_I = S \cap W_I$ and $S_J = S \cap W_J$.

Condition A. (1) $T_{W_I}X$ is square integrable with respect to π and $\int T_{W_I}X\pi(dX) = 0$. Further, $R \equiv \int T_{W_I}X \cdot (T_{W_I}X)'\pi(dX)$ is nondegenerate on W_I and satisfies $QR + RQ' = R$, where Q' is the transpose of Q.
(2) $T_{W_{J_1}}X$ is integrable with respect to π and $\int T_{W_{J_1}}X\pi(dX) = 0$.
(3) There exists a measure λ over S supported by S_J such that

(2.1) $\lim_{t \to \infty} t \cdot \pi(\{r^Q\theta\,; \theta \in F, r > t\}) = \lambda(F)$

holds for any Borel set F in S_J such that $\lambda(\partial F) = 0$.

Theorem 2.1. *Assume that real parts of eigen values of the exponent Q are all greater than or equal to $1/2$ and are not equal to 1. If Condition A is satisfied for the distribution $\pi = \log \nu$, then the distributions of $\varphi_t^{(n)}$ converge weakly. Let $\{\mu_t\}_{t>1}$ be the family of limit distributions. Then it is a convolution semigroup of stable distributions with respect to the dilation $\{\gamma_t\}_{t>0}$. Let L be the infinitesimal generator of the convolution semigroup. Then for any $f \in C^2$, Lf is represented by*

$$(2.2)\qquad Lf(\tau) = \frac{1}{2}\sum_{j,k} r_{jk}X_jX_kf(\tau) + \int_{\mathscr{G}-\{0\}} (f(\tau \exp X) - f(\tau) - T_{W_{I_1}} Xf(\tau))M(dX).$$

Here, $\{X_1, \ldots, X_d\}$ is a basis of $\mathscr{G}$, (r_{jk}) is the matrix representation of the covariance R with respect to the basis, and M is the measure over $\mathscr{G} - \{0\}$ defined by

$$(2.3)\quad M(E) = \int_S \lambda(d\theta) \int_{(0,\infty)} \chi_E(r^Q\theta)r^{-2}dr.$$

Proof. Define an array of G-valued random variables $\xi_{n,k}$, $n, k = 1,2,\ldots$ by $\xi_{n,k} = \gamma_{1/n}(\xi_k)$. Then $\xi_{n,k} = \exp(d\gamma_{1/n}\eta_k) = \exp((1/n)^Q\eta_k)$. For each fixed n, these are independent identically distributed random variables. We have $\varphi_t^{(n)} = \xi_{n,1}\cdots\xi_{n,[nt]}$, because $\gamma_t(\sigma\tau) = \gamma_t(\sigma)\gamma_t(\tau)$ is satisfied. In order to prove the weak convergence of distributions of $\varphi_t^{(n)}$, we shall apply a result in Kunita [2]. Let π_n be the distribution of $(1/n)^Q\eta_k$ over $\mathscr{G}$. We denote by M_n the restriction of the measure $n\pi_n$ to the subset $\mathscr{G} - \{0\}$. For a fixed $\delta > 0$, we define a linear transformation $A_{\delta,n}$ over $\mathscr{G}$ and a vector $B_{\delta,n} = (b^j_{\delta,n})$ as follows.

$$(2.4)\qquad A_{\delta,n} = n\int_{\{r(X)<\delta\}} X\cdot X'\pi_n(dX),$$

$$b^j_{\delta,n} = n\int_{\{r(X)<\delta\}} T_{W_j}X\pi_n(dX),$$

where $\{r(X) < \delta\} = \{X \in \mathscr{G}\,;\, r(X) < \delta\}$. We want to prove the following three:

(a) $\lim_{\delta\to 0}\limsup_{n\to\infty} \| A_{\delta,n} - R \| = 0$, where $\| \ \|$ is the norm of the linear transformation.

(b) The sequence of measures $\{M_n\}$ converges to M *vaguely* in the following sense.

$$\int_{\{r(X)<\varepsilon\}} f(X)M_n(dX) \to \int_{\{r(X)<\varepsilon\}} f(X)M(dX)$$

for any $0 < \varepsilon \le \infty$ and $f \in C_0(\pi)$. Here $C_0(\pi)$ is the set of all continuous functions f over $\mathscr{G} - \{0\}$ such that $\lim_{X\to 0}f(X) = 0$, $\lim_{X\to\infty}f(X)$ exists and $\left\{\int |f(X)\log|f(X)\|M_n(dX)\right\}$ is bounded.

(c) The sequence of the vectors $\{B_{\delta,n}\}$ converges for any $\delta > 0$.

If these three properties are verified, then the sequence of the distributions of $\varphi_t^{(n)}$, $n = 1, 2, \ldots$ converges weakly and the family of the limit distributions is a convolution semigroup by a slight modification of Theorem 3 in [2]. It is in fact stable with respect to the given dilation. The representation (2.2) of the infinitesimal generator is shown in [3].

We shall first prove (a). Since $\| A_{\delta,n} - R \| \le \| T_{W_I}A_{\delta,n}T'_{W_I} - R \| + 2\| T_{W_I}A_{\delta,n} \|$, it is sufficient to prove that each term of the right hand side converges to 0 as $n \to \infty$ and $\delta \to 0$. Consider first $T_{W_I}A_{\delta,n}T'_{W_I} - R$. The matrix $A_{\delta,n}$ is written by

$$A_{\delta,n} = (1/n)^{Q-(1/2)I}R_{\delta,n}(1/n)^{(Q-(1/2)I)'},$$

$$\text{where } R_{\delta,n} = \int_{\{r(X)<n\delta\}} X\cdot X'\pi(dX).$$

Let $R^{1/2}$ be a unique linear symmetric trasformation on $\mathscr{G}$ such that $(R^{1/2})^2 = R$, $R^{1/2}W_I = W_I$ and $R^{1/2}W_I^{\perp} = 0$, where $W_I^{\perp}$ is the orthogonal complement of W_I in $\mathscr{G}$. $R^{-1/2}$ is defined similarly. Then we have the equality

$$T_{W_I}A_{\delta,n}T'_{W_I} - R = R^{1/2}K_nR^{-1/2}(T_{W_I}R_{\delta,n}T'_{W_I} - R)R^{-1/2}K'_nR^{1/2},$$

where $K_n = R^{-1/2}(1/n)^{(Q-(1/2)I)}R^{1/2}$. It holds $\|K_n\| \le 1$ for all n. Indeed, the property $QR + RQ' = R$ implies $t^Q Rt^{Q'} = tR$ or $t^{Q-(1/2)I}Rt^{(Q-(1/2)I)'} = R$ for any $t > 0$, so that $K_nK'_n = T'_{W_I}$. See Proposition 4.3.3 in [1]. Now since $\| T_{W_I}R_{\delta,n}T'_{W_I} - R \| \to 0$ as $n \to \infty$, we obtain $\| T_{W_I}A_{\delta,n}T'_{W_I} - R \| \to 0$ as $n \to \infty$. We next consider $T_{W_I}A_{\delta,n}$. By (2.4), we have

$$\| T_{W_I}A_{\delta,n} \| \le \int_{S\times(0,\delta)} |T_{W_I}r^Q\theta|^2 G_n(d\theta dr),$$

where $G_n(F_1 \times F_2) = M_n(\{\theta(X) \in F_1, r(X) \in F_2\})$. Let q be the minimum of α_j such that $\alpha_j > 1/2$. Note that

$$(2.5)\qquad T_{W_I}r^Q\theta = \sum_{j\in J}\sum_{k=0}^{l_j-1} \frac{1}{k!}(\log r)^k \times \overline{(r^{\kappa_j}(Q-\kappa_j)^kT_j\theta + r^{\bar{\kappa}_j}(Q-\bar{\kappa}_j)^kT_j\theta)},$$

where T_j is the projector to $\operatorname{Ker}((Q-\kappa_j)^{l_j})$. Then for ε with $0 < \varepsilon < q - 1/2$, there is a positive constant c such that $|T_{W_I}r^Q\theta| \le cr^{q-\varepsilon}$ for all $\theta \in S$ and $r < 1$. Set $F_n(t) = G_n(S \times [t, \infty))$. Then $\| T_{W_I}A_{\delta,n} \|$ is dominated by

$$- c^2 \int_{(0,\delta)} r^{2(q-\varepsilon)} F_n(dr) \leq$$

$$c^2(\delta^{2(q-\varepsilon)} F_n(\delta) + 2(q-\varepsilon) \int_0^\delta t^{2(q-\varepsilon)-1} F_n(t)\,dt).$$

Since $F_n(t) = n\pi(\{r(X) > nt\})$, $\lim_{n\to\infty} F_n(t) = \lambda(S)t^{-1}$ holds for any $t > 0$ by Condition A (2). Therefore,

$$\limsup_{n\to\infty} \| T_{W_I} A_{\delta,n} \| \leq c^2\lambda(S)(\delta^{2(q-\varepsilon)-1} + 2(q-\varepsilon)\int_0^\delta t^{2(q-\varepsilon)-2} dt),$$

which tends to 0 as $\delta \to 0$ because $2(q-\varepsilon) > 1$. We have thus proved the assertion (a).

We shall next prove (b). For any $\delta > 0$, we have

$$M_n(\{\theta(X) \in S_I,\ r(X) \geq \delta\})$$
$$\leq \frac{n}{\delta^2} \int_{\{\theta(X)\in S_I, r(X)\geq\delta\}} | T_{W_I} X |^2 \pi_n(dX)$$
$$= \frac{1}{\delta^2} \mathrm{Tr}(R^{1/2} K_n R^{-1/2} (R - T_{W_I} R_{\delta,n} T'_{W_I}) R^{-1/2} K'_n R^{1/2}).$$

It tends to 0 as $n \to \infty$. Then $\int f(X) M_n(dX) \to 0$ as $n \to \infty$ if f is a function of the form $f(X) = f(T_{W_I} X)$. Now let F be a Borel set in S_I satisfying (2.1) and let

$$E = \{r^Q\theta : \theta \in F,\ a < r \leq b\}.$$

Then $\lim_{n\to\infty} M_n(E) = M(E)$. Indeed by Condition A (3), we have

$$\lim_{n\to\infty} M_n(E) = \int_F \lambda(d\theta) \left(\frac{1}{a} - \frac{1}{b}\right)$$
$$= \int_F \lambda(d\theta) \int_a^b \frac{1}{r^2}\, dr = M(E).$$

Then $\left\{\int f dM_n\right\}$ converges to $\int f dM$ if $f \in C_0(\pi)$ is of the form $f(X) = f(T_{W_I} X)$. Consequently, $\{M_n\}$ converges vaguely to M on $\mathscr{G} - \{0\}$.

Finally we shall prove (c). We first consider the case where $\alpha_j = 1/2$. Since the integral of $T_{W_j} X$ by M_n over $\mathscr{G}$ is 0 by Condition A(1), we have

$$(2.6) \qquad b^j_{\delta,n} = - \int_{\{r(X)\geq\delta\}} T_{W_j} X\, M_n(dX).$$

The domain $\{r(X) \geq \delta\}$ of the integral can be restricted to $\{\theta(X) \in S_I,\ r(X) \geq \delta\}$. Then by Schwarz's inequality,

$$| b^j_{\delta,n} | \leq \left(\int | T_{W_j} X |^2 M_n(dX)\right)^{1/2} \times M_n(\{\theta(X) \in S_I,\ r(X) \geq \delta\})^{1/2}.$$

The first term of the right hand side is bounded in n. The second term converges to 0. Therfore for any $\delta > 0$, $\lim_{n\to\infty} b^j_{\delta,n}$ exists and is equal to 0 if $j \in I$. Next consider the case $1/2 < \alpha_j < 1$. By Condition A(2), $b^j_{\delta,n}$ is written as (2.6). Note the equality (2.5). Then for $N > 0$, the truncated function $f_N = (T_{W_j} r^Q \theta \wedge N) \vee (-N)$ belongs to $C_0(\pi)$. Therefore the limit of $b^j_{\delta,n}$ exists and is equal to $-\int_\delta^\infty \int_{S_I} T_{W_j} r^Q \theta \lambda(d\theta) r^{-2} dr$ for any $\delta > 0$. In the case where $\alpha_j > 1$, we can show similarly that $\lim_{n\to 0} b^j_{\delta,n}$ exists and is equal to $\int_0^\delta \int_{S_I} T_{W_j} r^Q \theta \lambda(d\theta) r^{-2} dr$ for any $\delta > 0$. The proof is complete.

The following corresponds to a central limit theorem on the Lie group.

Corollary 2.2. *Assume that* $\eta_k \equiv \log \xi_k$ *is of mean* 0 *and has a finite nonsingular covariance* R *If* $A \equiv T_{W_I} R T'_{W_I}$ *satisfies* $QA + AQ' = A$, *the distributions of* $\varphi_t^{(n)}$ *converge weakly. Let* $\{\mu_t\}_{t>0}$ *be the family of limit distributions. Then it is a convolution semigroup of stable distributions with respect to the dilation* $\{\gamma_t\}_{t>0}$. *Further its characteristics are given by* $(A, 0, 0)$.

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