On a Borsuk-Ulam Theorem for Stiefel Manifolds

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§1. Introduction. The classical Borsuk-Ulam theorem states that if a continuous map $f: S^n \to R^n$ is $Z_2 = O(1)$-equivariant with the antipodal involutions, then $f^{-1}(0)$ is not empty. We consider G-spaces X, Y and a G-map $f: X \to Y$, i.e., continuous and G-equivariant. The purpose of this note is to extend Borsuk-Ulam theorem for a G-map.

Here, let $X = V_m(R^{m+n})$ be the Stiefel manifold, the space of orthonormal m-frames in R^{m+n}, and let $Y = (R^{m+k})^m$ be a space of m-tuples of vectors in R^{m+k}. Then we can regard $X = V_m(R^{m+n})$ and $Y = (R^{m+k})^m$ as the orthogonal group $O(m)$-spaces naturally. Now let $f: V_m(R^{m+n}) \to (R^{m+k})^m$ be an $O(m)$-map.

To generalize Borsuk-Ulam theorem, let us replace $\{0\}$ to the subspace of $(R^{m+k})^m$, denoted by $(R^{\widetilde{m+k}})^m$ consisting of all linearly dependent vectors in R^{m+k}. Note that $(R^{\widetilde{m+k}})^m$ is $O(m)$-invariant. Now take any $O(m)$-map $f: V_m(R^{m+n}) \to (R^{m+k})^m$.

In this note, we are concerned with the orbit space $A_f/O(m)$. For $m = 2$, the following theorem has been known (cf. [2; Theorem 5. 2]):

Theorem. *If $k < n$ and $f: V_2(R^{n+2}) \to (R^{k+2})^2$ is a map then $\dim(H^*(A_f/O(2))) \geq 2n - k - 2$, where we use the Alexander-Spanier cohomology with coefficients in Z_2.*

We generalize the above theorem as follows:

Theorem 1.1. *If $m \geq 2$ and $k < n$, then $H^l(A_f/O(m)) \neq 0$ for some $l \geq mn - k - m$.*

Furthermore we also obtain the following:

Theorem 1.2. (i) *If $m = 2$, $n = 2^s - 1$ and $k \neq 2^t - 1$, then $H^{2(n-k)}(A_f/O(2)) \neq 0$,* (ii) *If $m = 3$, $n = 2^s - 2$ and $k = 2^t - 2$, then $H^{2(n-k)-1}(A_f/O(3)) \neq 0$.* (iii) *If $m \geq 2$, $n = 2^s - m + 1$ and $k = 2^t - m$, then $H^l(A_f/O(m)) \neq 0$ for $l = 2n + m - k - 4$.*

Preparing a general theory of index in §2, we obtain the $O(m)$-index of the Stiefel manifold in §3. We prove Theorems 1.1 in §4 and 1.2 in §5.

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§2. Ideal valued index of a G-space. Let G be a compact Lie group, EG and BG be its universal and classifying spaces respectively. Then for any G-space X, denote by $EG \times_G X$ the orbit space of the diagonal G-action on $EG \times X$.

The index of X is given as follows:

(2.1) *The projection $p: EG \times_G X \to BG$ induces the homomorphism $H^*(BG) \to H^*(EG \times_G X)$. We set*

$$\mathrm{Ind}^G X = \mathrm{Ker}(p^*).$$

This index satisfies the following:

(2.2) ([2; Proposition 2.3]) *Let X and Y be G-spaces and $f: X \to Y$ be a G-map. Then*

$$\mathrm{Ind}^G X \supset \mathrm{Ind}^G Y.$$

(2.3) ([2; Theorem 2.4]) *Let X and Y be G-spaces and $\tilde{Y} \subset Y$ be a G-invariant closed subspace. Then*

$$\mathrm{Ind}^G f^{-1}(\tilde{Y}) \cdot \mathrm{Ind}^G(Y - \tilde{Y}) \subset \mathrm{Ind}^G X.$$

If the given G-action on X is free, then the projection $EG \times_G X \to X/G$ induces the isomorphism $H^*(X/G) \to H^*(EG \times_G X)$.

§3. The index of $O(m)$-spaces. In this section, we study the index of $O(m)$-spaces. The universal $O(m)$-spaces is the Stiefel manifold $V_m(R^\infty)$, and its orbit space is the Grassmann manifold $G_m(R^\infty)$. The cohomology ring of $G_m(R^\infty)$ is the polynomials $Z_2[w_1, \ldots, w_m]$ of the Stiefel Whitney classes $w_r \in H^r(G_m(R^\infty))$ $(1 \leq r \leq m)$. Thus we obtain the polynomials $\bar{w}_s \in H^s(G_m(R^\infty))$ $(s \geq 1)$ of $w_1, \ldots, w_m$ by the formula

$$(1 + w_1 + \cdots + w_m)(1 + \bar{w}_1 + \bar{w}_2 + \cdots) = 1.$$

Let $J(m, n)$ be the ideal of $H^*(G_m(R^\infty))$ generated by $\bar{w}_{1+n}, \ldots, \bar{w}_{m+n}$. The inclusion i:

$G_m(R^{m+n}) \to G_m(R^\infty)$ induces the epimorphism $i^*: H^*(G_m(R^\infty)) \to H^*(G_m(R^{m+n}))$, and its kernel is $J(m, n)$. Hence we have:

(3.1) [2; Theorem 3.3] $\mathrm{Ind}^{O(m)} V_m(R^{m+n}) = J(m, n)$.

Let $J(m, n)_r = J(m, n) \cap H^r(G_m(R^\infty))$, and

$$\gamma(r, n): \bigoplus_{1 \le s \le m} H^{r-n-s}(G_m(R^\infty)) \to H^r(G_m(R^\infty))$$

be a homomorphism given by

$$\gamma(r, n)(x_1, \dots, x_m) = \bar{w}_{1+n}x_1 + \cdots + \bar{w}_{m+n}x_m$$

for $x_s \in H^{r-n-s}(G_m(R^\infty))$. Then

(3.2) [2; Lemma 3.5] $\mathrm{Im}\gamma(r, n) = J(m, n)_r$.

Let $f: V_m(R^{m+n}) \to (R^{m+k})^m$ be an $O(m)$-map, and set $(R^{m+k})^m_0 = (R^{m+k})^m - (\widehat{R^{m+k}})^m$. By [4; Lemma 3.1], $(R^{m+k})^m_0$ is $O(m)$-equivariantly deformable to $V_m(R^{m+k})$, and hence $\mathrm{Ind}^{O(m)}(R^{m+k})^m_0 = \mathrm{Ind}^{O(m)} V_m(R^{m+k})$. Therefore the following holds by (2.3) and (3.1):

(3.3) [2; Theorem 4.1] $\mathrm{Ind}^{O(m)} A_f \cdot J(m, k) \subset J(m, n)$.

§4. The proof of Theorem 1.1. In this section we set $r = mn$. Then we have the following:

Lemma 4.1. (i) $J(m, n)_r \subset H^r(G_m(R^\infty))$.
(ii) $\gamma(r, k): \bigoplus_{1 \le s \le m} H^{r-k-s}(G_m(R^\infty)) \to H^r(G_m(R^\infty))$ *is surjective for* $k < n$.

Proof. Since $\dim G_m(R^{m+n}) = r$, we have $H^r(G_m(R^\infty))/J(m, l)_r \cong H^r(G_m(R^{m+l})) \cong Z_2$ for $l = n$ and 0 for $l < n$. Hence (i) holds, and (ii) follows immediately from (3.2).

Q.E.D.

Proof of Theorem 1.1. By the above lemma we can choose $(x_1, \dots, x_m) \in \bigoplus_{1 \le s \le m} H^{r-k-s}(G_m(R^\infty))$ satisfying $\gamma(r, k)(x_1, \dots, x_m) \notin J(m, n)$.

Assume that $x_s \in \mathrm{Ind}^{O(m)} A_f$ for any $1 \le s \le m$. By (3.3) we get

$$\gamma(r, k)(x_1, \dots, x_m) \in \mathrm{Ind}^{O(m)} A_f \subset J(m, n).$$

This contradicts the first condition. Therefore $x_s \notin \mathrm{Ind}^{O(m)} A_f$ for some $1 \le s \le m$, and hence $H^{r-k-s}(A_f/O(m)) \ne 0$.

Thus the proof of the theorem is completed

Q.E.D.

§5. The proof of Theorem 1.2. By using the results of H. Hiller [3], we show Theorem 1. 2.

Proof of Theorem 1.2. (i) In the case $m = 2$, $n = 2^s - 1$ and $k \ne 2^t - 1$. By [3; Theorems 2.3 and 3.3], $w_1^{2k} \in J(2, k)$ and $w_1^{2n} \notin J(2, n)$. From (3.3) it follows that $w_1^{2n-2k} \notin \mathrm{Ind}^{O(2)} A_f$, and hence $H^{2n-2k}(A_f/O(2)) \ne 0$.

(ii) In the case $m = 3$, $n = 2^s - 2$ and $k = 2^t - 2$. By [3; Theorem 3.3 and Lemma 4.6], $w_1^{2n+2} \notin J(3, n)$ and $w_1^{2k+3} \in J(3, k)$. Thus $H^{2n-2k-1}(A_f/O(3)) \ne 0$.

(iii) In the case $m \ge 2$, $n = 2^s - m + 1$ and $k = 2^t - m$. By [3; Proposition 2.2 and Theorem 3.3], $w_1^{k+m} \in J(m, k)$ and $w_1^{2(n+m-2)} \notin J(m, n)$. Then $H^{2n+m-k-4}(A_f/O(m)) \ne 0$. Terefore the proof of the theorem is completed.

Q.E.D.

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