# Radon Transform on Distributions

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**Abstract**: In the literature there are three apparently different definitions of the Radon transform on various spaces of distributions: Gelfand-Graev's, Helgason's and Ludwig's. In this paper a new definition of the Radon transform on the space of the tempered distributions is given and it is proved that, properly understood, the earlier definitions are all equivalent to the new one. A constructive description (a characterization) of the space of test functions is given. A simple method for studying the range of the Radon transform on some spaces of distributions is described.

**Key words**: Tomography; Radon transform; distributions.

**1. Introduction.** In the literature there are three apparently different definitions of the Radon transform $R$ on the spaces of distributions. The first, given by I. Gelfand and M. Graev, (GG definition), (see [1]), Helgason's definition (H definition) based on the duality formula ([2]), and Ludwig's definition (L definition) ([3]).

In this paper these definitions are analyzed, their common features are demonstrated; a new definition is given and it is proved that, properly understood, the earlier definitions are all equivalent to the new one and, therefore, they are all equivalent; a constructive description of the space of test functions is given; a simple method for studying the range of $R$ on distributional spaces is described.

We use the notations and some results from the forthcoming monograph [4]. In [5]-[6] some properties of the Radon transform on various spaces of distributions are given and H definition is used. In [7] and [8] some related results are obtained.

**2. The three definitions of $R$.** 2.1. Let us introduce the standard notations: $\mathcal{S}' := \mathcal{S}'(\boldsymbol{R}^n)$ is the space of tempered distributions on $\mathcal{S} := \mathcal{S}(\boldsymbol{R}^n)$, $\mathcal{D}'$ is the space of distributions on $\mathcal{D} := C_0^\infty(\boldsymbol{R}^n)$, and $\mathcal{E}'$ is the space of distributions on $\mathcal{E} := C^\infty(\boldsymbol{R}^n)$. The Radon transform $R$ on $\mathcal{S}$ is defined by the formula:

$$Rf := \tilde{f} := \int_{l_{\alpha p}} f(x)\,ds, \tag{1}$$

where $l_{\alpha p} = \{x : \alpha \cdot x = p\}$ is a plane, $ds$ is the Lebesgue measure on this plane, $p \in \boldsymbol{R}$, $\alpha \in S^{n-1}$, the unit sphere in $\boldsymbol{R}^n$.

Let $\tilde{f} := \mathcal{F} f := \int_{\boldsymbol{R}^n} \exp(ix \cdot \xi) f(x)\,dx$, $Fh := \int_{-\infty}^{\infty} \exp(ip\lambda) h(\alpha, p)\,dp$, $\xi = \lambda\alpha$. An elementary corollary of the Fourier slice theorem is [4]:

$$R = F^{-1}\mathcal{F} \text{ on } \mathcal{S}. \tag{2}$$

Let $\mathcal{S}_e := \mathcal{S}_e(Z)$ be the Schwartz space of even functions $h(-\alpha, -p) = h(\alpha, p)$ on $Z := \mathcal{S}^{n-1} \times \boldsymbol{R}$. If $R$ is considered as an operator from $\mathcal{S}$ into $\mathcal{S}_e(Z)$, then its range is the subspace $\mathcal{S}_{em} \subset \mathcal{S}_e$ which consists precisely of the functions $h \in \mathcal{S}_e(Z)$ satisfying the moment conditions

$$\int_{-\infty}^{\infty} h(\alpha, p) p^k\,dp = \mathcal{P}_k(\alpha), \quad k = 0,1,2,\ldots, \tag{3}$$

where $\mathcal{P}_k(\alpha)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a restriction on $\mathcal{S}^{n-1}$ of a homogeneous polynomial of degree $k$ defined on $\boldsymbol{R}^n$. The well known result is: $\mathcal{S}_{em} = R\mathcal{S}$.

Let $\langle f, \phi \rangle$ denote the $L^2(\boldsymbol{R}^n)$ inner product, and $(g, h)$ denote the $L^2(Z)$ inner product. Let $R^*$ be the adjoint operator

$$R^* h = \int_{S^{n-1}} h(\alpha, \alpha \cdot x)\,d\alpha. \tag{4}$$

**2.2.** The GG definition is based on the Parseval formula for the Radon transform:

$$\langle f, \phi \rangle = (\tilde{f}, \Phi) \tag{5}$$

where

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(6) $\Phi := \frac{(-i)^n}{(2\pi)^n} \partial_p^{n-1} \frac{1}{p - i0} \circledast \tilde{\phi}, \ \partial_p^{n-1} := \frac{\partial^{n-1}}{\partial p^{n-1}},$
and $\circledast$ denotes the convolution with respect to $p$-variable. One can check that

(7) $\quad (\tilde{f}, \Phi) = (\tilde{f}, KR\phi) \quad \forall f, \phi \in \mathcal{S},$

where [4]

$$(8) \quad K := \begin{cases} \gamma(-1)^{\frac{n-1}{2}} \partial_p^{n-1}, & \text{if } n \text{ is odd}, \\ \gamma(-1)^{\frac{n}{2}} \mathcal{H}\partial_p^{n-1}, & \text{if } n \text{ is even}, \end{cases} \qquad \gamma := \frac{1}{2(2\pi)^{n-1}},$$

and

$$(9) \quad \mathcal{H}h := -\frac{1}{\pi p} \circledast h := -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{h(\alpha, q)\,dq}{p - q}, \qquad F\mathcal{H}h = -i \operatorname{sgn} \lambda F h$$

Let $f \in \mathcal{S}'$. Choose $f_n \in \mathcal{S}$, $f_n \to f$ in $\mathcal{S}'$. By passing to the limit, equation (7) is verified for any $f \in \mathcal{S}'$. One can prove that [4]:

(8a) $\quad K = \gamma F^{-1} |\lambda|^{n-1} F.$

For any $f \in \mathcal{S}'$, the left-hand side of (5) is a bounded linear functional on $\mathcal{S}$. The GG definition of $\tilde{f}$, based on formula (5), defines $\tilde{f}$ on the set of the test functions $\Phi$ of the form (6), where $\phi$ runs through $\mathcal{S}$. A description of this set for odd $n$ is given in [1] and it is claimed that the case of even $n$ can be treated similarly. There is, however, an essential difference in the rate of decay of the test functions: if $n$ is odd, the decay is faster than any negative power of $p$ as $|p| \to \infty$; if $n$ is even, the decay is, generally, as $O(|p|^{-n})$.

**2.3.** The H definition is based on the duality formula:

(9) $\quad (Rf, h) = \langle f, R^*h \rangle$

This formula is used in [2] to define $Rf$ for $f \in \mathcal{E}'$, but not for $f \in \mathcal{S}'$. The reason is simple: for $h \in \mathcal{S}_e$, the function $R^*h$ may decay not faster than $O(|x|^{-1})$ as $|x| \to \infty$. Therefore $R^*h$ is not, in general, in the space $\mathcal{S}$, so that the right-hand side of (9) is not well-defined for $f \in \mathcal{S}'$ and an arbitrary $h \in \mathcal{S}_e$.

**2.4.** The L definition [3] is based on the duality formula (9) as well, but the space of the test functions $h$ is not $\mathcal{S}_e$, as in [2]. This space in [3] is $KR\mathcal{S}$, where $K$ is defined in (8). One has an identity

(10) $\quad R^*KR = I$ on $\mathcal{S}$,

which is an immediate consequence of the inversion formula for $R$ on $\mathcal{S}$. Therefore, if $h = KR\phi$, $\phi \in \mathcal{S}$, then $R^*KR\phi = \phi$, so that the right-hand side of (9) is a well defined bounded linear functional on $\mathcal{S}$. Thus, for any $f \in \mathcal{S}$, formula (9) defines $\tilde{f}$ on the space $KR\mathcal{S} := \mathcal{S}_t$ of the test functions. If in H definition we change the space of the test functions, we get L definition. Therefore H and L definitions are applicable to $\mathcal{S}'$ if the space of test functions $h$ is $\mathcal{S}_t$, and H and L definitions are identical in the case.

**2.5.** Let us prove that the GG definition is equivalent to the H and L definitions. What we have to prove is formula (7).

**Lemma 1.** *Formula (7) holds.*

*Proof.* One has $(p - i0)^{-1} = p^{-1} + i\pi\delta(p)$. Thus, formula (6) yields

$$(\tilde{f}, \Phi) = \left(\tilde{f}, \frac{(-i)^n}{(2\pi)^n} \partial_p^{n-1}(-\pi\mathcal{H}\tilde{\phi} + i\pi\tilde{\phi})\right) = \begin{cases} \gamma(-1)^{\frac{n-1}{2}}(\tilde{f}, \partial_p^{n-1}R\phi), & n \text{ odd}, \\ \gamma(-1)^{\frac{n}{2}}(\tilde{f}, \mathcal{H}\partial_p^{n-1}R\phi), & n \text{ even}, \end{cases}$$

where we have used the formula $\partial_p\mathcal{H} = \mathcal{H}\partial_p$, and the evenness of $\tilde{f}$. Indeed, if $n$ is odd, then $(\tilde{f}, \mathcal{H}\partial_p^{n-1}R\phi) = 0$, while if $n$ is even, then $(\tilde{f}, \partial_p^{n-1}R\phi) = 0$. The proof is complete. □

**Remark 1.** Note that equation (8a) can be written as

(11) $\quad FKF^{-1} = \gamma|\lambda|^{n-1},$

where $\gamma$ is the same as in (8), and the right-hand side of (11) is an operator of multiplication by $\gamma|\lambda|^{n-1}$. It follows from formulas (5) and (7) that the GG definition is identical to L definition. Let us summarize the results.

**Theorem 1.** *The GG, H, and L definitions of $R$ on $\mathcal{S}'$ are equivalent.*

**3. The new definition.** We want to define $R$ on $\mathcal{S}'$ using formula (2), The reasons are:

(1) this definition is very convenient for actual calculations of $Rf$;

(2) it allows us to study the range of $R$ easily.

**Definition 1.** *Let $f \in \mathcal{S}'$. Then $\tilde{f} := Rf$ is defined by the formula*

(12) $\quad (\tilde{f}, h) = \frac{1}{\pi}\langle f, |\lambda|^{1-n} Fh \rangle \quad \forall h \in \mathcal{S}_t.$

The motivation is simple. By Parseval's formula for $f \in \mathcal{S}$ and $h \in \mathcal{S}_e$ one has

$$(\tilde{f}, h) = \frac{1}{2\pi}(F\tilde{f}, Fh) = \frac{1}{2\pi}(\tilde{f}(\lambda\alpha), Fh) = \frac{1}{2\pi}(\tilde{f}(\lambda\alpha)|\lambda|^{n-1}, |\lambda|^{1-n}Fh) = \frac{1}{\pi}\langle \tilde{f}, |\lambda|^{1-n}Fh \rangle.$$

One can also write (12) in the form

(12') $(\tilde{f}, h) = \gamma^{-1}\langle f, \mathscr{F}^{-1}|\lambda|^{1-n}Fh\rangle.$

Therefore

(12") $R^* = \gamma^{-1}\mathscr{F}^{-1}|\lambda|^{1-n}F.$

Thus (12) holds for $f \in \mathscr{S}$ and $h \in \mathscr{S}_e$. If $f \in \mathscr{S}'$, then the right-hand side of (12) is a bounded linear functional on the space $\mathscr{S}_t$, where $\mathscr{S}_t$ is the space of such $h$ for which

(13) $|\lambda|^{1-n}F_{p\to\lambda}h \in \mathscr{S}.$

We have denoted this space by the same letter as the space $\mathscr{S}_t := KR\mathscr{S}$ defined earlier. Let us prove that so defined $\mathscr{S}_t$ is identical with $KR\mathscr{S}$.

**Theorem 2.** *Formula* (13) *holds iff* $h \in KR\mathscr{S}$.

*Proof.* Sufficiency: if $h \in KR\mathscr{S}$, then $h = K\tilde{\phi}$, where $\phi \in \mathscr{S}$. Thus, $|\lambda|^{1-n}Fh = |\lambda|^{1-n}FK\tilde{\phi} = |\lambda|^{1-n}FKF^{-1}\tilde{\phi} = \gamma\tilde{\phi} \in \mathscr{S}$. Thus, $h \in \mathscr{S}_t$. Here we have used formula (11), which implies $|\lambda|^{1-n}FKF^{-1}\tilde{\phi} = \gamma\tilde{\phi}$.

Necessity: if $h \in \mathscr{S}_t$, then $|\lambda|^{1-n}Fh := \tilde{\phi} \in \mathscr{S}$ and $\phi := \mathscr{F}^{-1}\tilde{\phi} \in \mathscr{S}$, since $\mathscr{F}: \mathscr{S} \to \mathscr{S}$ is an isomorphism. Thus

$$h = F^{-1}|\lambda|^{n-1}\mathscr{F}\phi = F^{-1}|\lambda|^{n-1}FF^{-1}\mathscr{F}\phi$$
$$= F^{-1}|\lambda|^{n-1}F\tilde{\phi} = \gamma K\tilde{\phi} \in KR\mathscr{S}.$$

Here formulas (2) and (8a) were used. Theorem 2 is proved. □

**Remark 2.** Note that $\frac{1}{\gamma}|\lambda|^{1-n}Fh = \tilde{\phi}$ if $h = KR\phi$. The conclusion of Theorem 2 follows from this equation as well.

As an immediate consequence of Theorem 2 we obtain

**Theorem 3.** *Definition* 1 *is equivalent to the GG, H, and L definitions of* $R$ *on* $\mathscr{S}'$.

*Proof.* By Theorem 1 it is sufficient to check that Definition 1 is equivalent to the H definition. Let $h \in \mathscr{S}_t$. By Theorem 2 and formulas (12') and (12") one has:

$$\frac{1}{\pi}\langle\tilde{f}, |\lambda|^{1-n}Fh\rangle = \gamma^{-1}\langle f, \mathscr{F}^{-1}|\lambda|^{1-n}Fh\rangle$$
$$= \langle f, R^*h\rangle.$$

Theorem 3 is proved. □

**Remark 3.** Definition 1 and Theorem 3 can be considered a justification of formula (2) as a definition of $R$ on distributions. Indeed, (12) can be written as $(Rf, h) = \frac{1}{2\pi}(\mathscr{F}f, Fh) = (F^{-1}\mathscr{F}f, h)$ by Parseval's formula or by the formula $F^* = 2\pi F^{-1}$, where $F^*$ is the adjoint operator to $F$ with respect to the inner product $(\cdot, \cdot)$. The operator $R^{-1} = \mathscr{F}^{-1}F$ is well defined on $R\mathscr{S} = \mathscr{S}_{em}$.

**Remark 4.** The following important problem was not discussed in the literature: suppose

(A) $f \in \mathscr{S}' \cap C_{loc}(\boldsymbol{R}^n)$ and

$$\int_{l_{\alpha p}} |f(x)|\,ds < \infty \quad \forall \alpha, p,$$

so that formula (1) defines the classical Radon transform $R_cf$ of $f$.

Is it true that $R_cf = Rf$, where $Rf$ is understood as in Definition 1, or in the GG, H or L definitions?

This problem is important because it is known (see [4, section 2.7.3]) that there exists an $f \not\equiv 0$, $f \in C^\infty(\mathbf{R}^2)$, for which $R_cf \equiv 0$ (L. Zalcman's example). Clearly, at this $f$ either $Rf$ in the sense of Definition 1, is not defined, so that $f \notin \mathscr{S}'$, or $Rf \neq R_cf$, otherwise, by the injectivity of $R$, one has to conclude that $f = 0$, while, in fact, $f \not\equiv 0$. If, in addition to (A), we assume that $f \in L^1(\boldsymbol{R}^n)$ or, less restrictively, that $(1 + |x|)^{-1} f(x) \in L^1(\boldsymbol{R}^n)$, then we can prove that $R_cf = Rf$. It is an open problem to find the maximal set of $f$ on which the last equation holds.

**4. A constructive description of the space of test functions.** Let $\mathscr{S}_{sm}$ be a subspace of $\mathscr{S}(Z)$, which consists of the functions, whose parity is opposite to that of $n$, satisfying the shifted moment conditions:

$$(14)\quad \int_{-\infty}^{\infty} h(\alpha, p)p^k\,dp = \begin{cases}\mathscr{P}_{k+1-n}(\alpha), & k \geq n-1, \\ 0, & 0 \leq k < n-1,\end{cases}$$

where $\mathscr{P}_m(\alpha)$ is a restriction to $S^{n-1}$ of a homogeneous polynomial of degree $m$ of the variables $(\alpha_1, \ldots, \alpha_n)$ and the parity of $h(\alpha, p)$ is opposite to that of $n$. Define $\mathscr{S}_{\mathscr{H}} := \mathscr{H}\mathscr{S}_{sm}$ where $\mathscr{H}$ is defined in (9).

**Theorem 4.** *One has is*

$$(15)\quad \mathscr{S}_t = \begin{cases}\mathscr{S}_{sm}, & \textit{if } n \textit{ is odd}, \\ \mathscr{S}_{\mathscr{H}}, & \textit{if } n \textit{ is even}.\end{cases}$$

*Proof.* One has

(16) $\mathscr{S}_t = KR\mathscr{S} = K\mathscr{S}_{em}.$

Let $n$ be odd. Then, by formula (8), $K$ is proportional to $\partial_p^{n-1}$. Thus

(17) $\mathscr{S}_t = \partial_p^{n-1}\mathscr{S}_{em}.$

The right-hand side of (17) is precisely the space $\mathscr{S}_{sm}$, because the derivative of order $n-1$ with respect to $p$ of a function $h(\alpha, p)$, which satisfies the moment conditions (3), is a function satisfying the shifted moment conditions (14). Conversely, if $h \in \mathcal{S}_e$ and (14) hold, then $\partial_p^{-(n-1)} h \in \mathcal{S}_{em}$, where $\partial_p^{-(n-1)} h := \frac{1}{(n-2)!} \int_{-\infty}^{p} (p-q)^{n-2} h(\alpha, q) dq$. This claim can be easily checked using conditions (14) and integrating by parts.

If $n$ is even, then (16) and (8) yield:

$$\mathcal{S}_t = \mathcal{H}\mathcal{S}_{sm} := \mathcal{S}_{\mathcal{H}} \tag{18}$$

Theorem 4 is proved. □

**5. Range theorems.** Suppose $g(\alpha, p) \in \mathcal{S}'_e(Z)$. When does there exist an $f \in \mathcal{S}'$ such that $Rf = g$?

By Definition 1, formula (2) holds, so one has to calculate $f := \mathcal{F}^{-1}Fg$ and check if $f \in \mathcal{S}'$. Alternatively, $(g, h) = \frac{1}{\pi} \langle Fg, |\lambda|^{1-n} Fh \rangle$, $\forall h \in \mathcal{S}_t$, so that $g = Rf$ for some $f \in \mathcal{S}'$ iff $Fg \in \mathcal{S}'(\mathbf{R}^n_\xi)$, $\xi = \lambda\alpha$.

Recall that an entire function $\tilde{f}(\xi)$ is of exponential type $\le a$ iff $|\tilde{f}(\xi)| \le c \exp[(a+\varepsilon)|\xi|]$ for any $\varepsilon > 0$.

Let $\mathcal{D}'_\alpha(R)$ be the subset of $\mathcal{D}'_e(Z)$ which consists of distributions $g(\alpha, p)$ such that for any $\alpha \in S^{n-1}$ the distribution $g(\alpha, p) \in \mathcal{D}'(\mathbf{R})$. Denote $B_a := \{x : |x| \le a\}$.

**Theorem 5.** *Let $g(\alpha, p) \in \mathcal{D}'_\alpha(\mathbf{R})$ and assume that $F_{p\to\lambda} g$ is an entire function of the variable $\xi = \lambda\alpha$ of exponential type $\le a$. Let $f := \mathcal{F}^{-1} F_{p\to\lambda} g$. Then $f \in \mathcal{E}'$, $\mathrm{supp} f \subseteq B_a$ and $Rf = g$. Conversely, if $f \in \mathcal{E}'$ and $g := Rf$, then $\tilde{f}(\xi) = F_{p\to\lambda} g$ is an entire function of exponential type $\le a$ and $g = F^{-1}_{\lambda\to p} \mathcal{F} f \in \mathcal{D}'_\alpha(\mathbf{R})$.*

*Proof.* If $g \in \mathcal{D}'_\alpha(\mathbf{R})$, then $f(x) := \mathcal{F}^{-1} F_{p\to\lambda} g$ is well defined. By the Paley-Wiener-Schwartz theorem, $f \in \mathcal{E}'$, $\mathrm{supp}\, f \subseteq B_a$ and $g = F^{-1}_{\lambda\to p}\mathcal{F} f = Rf$.

Conversely, if $f \in \mathcal{E}'$ and $\mathrm{supp} f \subseteq B_a$, then $\mathcal{F} f$ is an entire function of exponential type $\le a$, $g := F^{-1}\mathcal{F} f \in \mathcal{D}'_\alpha(\mathbf{R})$, and $g = Rf$. □

One can use the above method to study the range of $R$ on other spaces of distributions. If $g \in Y$, where $Y$ is some space of distributions or functions, then $f \in X$, where $X = \mathcal{F}^{-1}FY$, provided that the operator $\mathcal{F}^{-1}F$ is well defined on $Y$. In [5] a different characterization of the range of $R$ on distributional spaces is given.

**6. Examples.** The results in the following examples are new: although the two examples were mentioned in [1, p.71], the results were given without proofs and only for the odd $n$. Our result for the first example and odd $n$ agrees with the one in [1, p.71], while our formulas, proved in Example 2, differ from formulas 4 and 5 in [1, p.71].

**Example 1.** Let us consider $f(x) = 1$, $x \in \mathbf{R}^n$, $\tilde{f} = (2\pi)^n \delta(\xi)$. If $n$ is odd then $h \in \mathcal{S}_{sm}$, $|\lambda|^{1-n} = \lambda^{1-n}$ and, using conditions (14), one gets:

$$\begin{aligned}
(R1, h) &= \gamma^{-1} \langle \delta(\xi), |\lambda|^{1-n} Fh \rangle \\
&= \gamma^{-1} \langle \delta(\xi), \lambda^{1-n} Fh \rangle \\
&= \gamma^{-1} \langle \delta(\xi), \sum_{m=n-1}^{N} \frac{i^m}{m!} \mathcal{P}_{m+1-n}(\xi) + O(|\xi|^{N+2-n}) \rangle \\
&= \frac{\gamma^{-1} i^{n-1}}{(n-1)!} c,
\end{aligned}$$

$$c = \int_{-\infty}^{\infty} h(\alpha, p) p^{n-1} dp, \ \xi = \lambda\alpha,$$

where $\gamma$ is defined in (8).

Thus, if $n$ is odd, one gets

$$R1 = \frac{\gamma^{-1} i^{n-1} p^{n-1} a(\alpha)}{(n-1)!},$$

where $a(\alpha)$ is an arbitrary smooth even function $a(\alpha) = a(-\alpha)$, such that $\int_{S^{n-1}} a(\alpha) d\alpha = 1$. The evenness of $a(\alpha)$ is an a priori assumption motivated by the evenness of the members of the range of $R$ on $\mathcal{S}$. If one imposes a priori the requirement that the members of the range of $R$ should be homogeneous of degree $-1$, then the corresponding homogeneity requirement should be imposed on $a(\alpha)$. We see that $R1$ is not uniquely defined.

If $n$ is even, then $|\lambda|^{1-n} = \lambda^{1-n}\,\mathrm{sgn}\lambda$, $h = \mathcal{H}g$, $g \in \mathcal{S}_{sm}$, $Fh = -i\,\mathrm{sgn}\lambda Fg$, and one gets

$$(R1, h) = -\frac{\gamma^{-1} i^n}{(n-1)!} \int_{-\infty}^{\infty} (\mathcal{H}^{-1}h)(\alpha, p) p^{n-1} dp. \tag{19}$$

Therefore, for $n$ even, the result is:

$$R1 = \frac{-\gamma^{-1} i^n p^{n-1} a(\alpha)}{(n-1)!} \mathcal{H}, \tag{20}$$

where the action of the distribution (20) on a test function is defined by (19) and $a(\alpha)$ is described above.

**Example 2.** Let $f(x) = \delta(x_1, x_2, \ldots, x_k)$, $1 \le k < n$. Then

$$\tilde{f}(\xi) = (2\pi)^{n-k} \delta(\xi_{k+1}, \ldots, \xi_n),$$

and Definition 1 yields for real-valued $h$:

$$(Rf, h) = \frac{(2\pi)^{n-k}}{2\pi} \int_{S^{n-1}} d\alpha \int_{-\infty}^{\infty} d\lambda\, \delta(\xi_{k+1}, \ldots, \xi_n) \int_{-\infty}^{\infty} dp \exp(-ip\lambda) h(\alpha, p)$$

$$= \int_Z d\alpha dp h(\alpha, p)(2\pi)^{n-k-1} \int_{-\infty}^{\infty} d\lambda \exp(-i\lambda p)\delta(\lambda\alpha_{k+1}) \cdot \ldots, \delta(\lambda\alpha_n).$$

Thus

(21) $$Rf = (2\pi)^{n-k-1}\delta(\alpha_{k+1}, \ldots, \alpha_n) \int_{-\infty}^{\infty} |\lambda|^{-n+k} \exp(-i\lambda p)d\lambda,$$

To calculate this integral, we use the formulas:

(22) $$F(|\lambda|^{-2m-1}) = c_0^{(2m+1)}p^{2m} - c_{-1}^{(2m+1)}p^{2m}\,\text{In}\,|p|,$$

$$F(|\lambda|^{-2m}) = (-1)^m \frac{\pi}{(2m-1)!}|p|^{2m-1},$$

where the coefficients $c_q^{(2m+1)}$ are defined by the expansion:

$$-2\sin\frac{\lambda\pi}{2}\Gamma(\lambda+1) = \frac{c_{-1}^{(n)}}{\lambda+n} + c_0^{(n)} + \ldots.$$

For example, let $n = 2, k = 1$. Then $\tilde{f} = 2\pi\delta(\xi_2)$, $m = 0$, and

$$(Rf, h) = \int_Z d\alpha dp h(\alpha, p)\delta(\alpha_2)\int_{-\infty}^{\infty}|\lambda|^{-1} \exp(-i\lambda p)d\lambda$$

$$= \int_Z d\alpha dp h\delta(\alpha_2)\cdot(c_0^{(1)} - c_{-1}^{(1)}\text{In}\,|p|).$$

Thus, for $n = 2$ and $k = 1$,

$$R\delta(x_1) = \delta(\alpha_2)(c_0^{(1)} - c_{-1}^{(1)}\text{In}\,p).$$

## References

[1] Gelfand, I., Graev, M., and Vilenkin, N.: Integral Geometry and Representation Theory. Academic. Press, New York (1965).

[2] Helgason, S.: The Radon Transform. Birkhauser, Boston (1980).

[3] Ludwig D.: The Radon transform on Euclidean spaces. Comm. Pure Appl. Math., **19**, 49–81 (1966).

[4] Ramm A. G. and, Katsevich A. I.: Radon transform and its applications (to appear).

[5] Hertle A.: On the range of Radon transform and its dual. Math. Ann., **267**, 91–99 (1984).

[6] Hertle A.: Continuity of the Radon transform and its inverse on Euclidean spaces. Math. Z., **184**, 165–192 (1983).

[7] Ramm A. G.: The Radon transform is an isomorphism. Appl. Math. Lett., **8**, 25–29 (1995).

[8] Ramm A. G.: Inversion formulas for the backprojection operator in tomography. Proc. Amer. Math. Soc. (to appear).