On Jeśmanowicz' Conjecture Concerning Pythagorean Numbers*),**)

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Abstract: Let r, s be positive integers satisfying $r > s$, $2 \mid r$ and $\gcd(r, s) = 1$. In this paper, using Baker's method, we prove that if $2 \| r$, $r \geq 81s$ and $s \equiv 3 \pmod 4$, then the equation $(r^2 - s^2)^x + (2rs)^y = (r^2 + s^2)^z$ has the only solution $(x, y, z) = (2,2,2)$.

Key words and phrases: Exponential diophantine equation; Jeśmanowicz' conjecture; Baker's method.

1. Introduction. Let $\boldsymbol{Z}$, $\boldsymbol{N}$, $\boldsymbol{Q}$ be the sets of integers, positive integers and rational numbers, respectively. Let (a, b, c) be a primitive Pythagorean triple such that

(1) $$a^2 + b^2 = c^2,\ a, b, c \in \boldsymbol{N},\quad \gcd(a, b, c) = 1,\ 2 \mid b.$$

Then we have, as is well known,

(2) $$a = r^2 - s^2,\ b = 2rs,\ c = r^2 + s^3,$$

where r, s are positive integers satisfying $r > s$, $\gcd(r, s) = 1$ and $2 \mid rs$. In [2], Jeśmanowicz conjectured that the only solution of the equation

(3) $$a^x + b^y = c^z,\quad x, y, z \in \boldsymbol{N}$$

is $(x, y, z) = (2,2,2)$. This conjecture was proved for some special cases (see the references of [4]). But, in general, the problem is not solved as yet. Recently, Takakuwa and Asaeda [6] proved that if $2 \| r$, $s = 3$ and r satisfies some other conditions, then the only solution of (3) is $(x, y, z) = (2,2,2)$. Guo and Le [1] showed that the conditions on r can be reduced to $r \geq 6000$, improving the result of [6]. In this paper we prove a general result as follows:

Theorem. If $2 \| r$, $s \equiv 3 \pmod 4$ and $r \geq 81s$, then the only solution of (3) is $(x, y, z) = (2,2,2)$.

By this theorem, the above condition $r \geq 6000$ in the result of [1] can be replaced by $r \geq 243$.

2. Preliminaries. Lemma 1. ([5, page 2]). The equation

$$X^4 + Y^2 = Z^4,\ X, Y, Z \in \boldsymbol{N}$$

has no solution (X, Y, Z).

Lemma 2. ([1, Lemma 2]). Let (x, y, z) be a solution of (3) with $(x, y, z) \neq (2,2,2)$. If $2 \| r$ and $s \equiv 3 \pmod 4$, then we have $2 \mid x$, $y = 1$ and $2 \nmid z$.

Let α be a nonzero algebraic number with the defining polynomial $a_0 z^n + a_1 z^{n-1} + \cdots + a_n = a_0(z - \sigma_1\alpha)\cdots(z - \sigma_n\alpha)$, where $a_0 \in \boldsymbol{N}$, $\sigma_1\alpha, \cdots, \sigma_n\alpha$ are all the conjugates of α. Then

$$h(\alpha) = \frac{1}{n}\Big(\log a_0 + \sum_{i=1}^{n} \log \max(1, |\delta_i\alpha|)\Big)$$

is called Weil's height of α.

Lemma 3. Let α_1, α_2 be positive real algebraic numbers which are multiplicatively independent. Further let $D = [\boldsymbol{Q}(\alpha_1, \alpha_2) : \boldsymbol{Q}]$ and $\log A_j = \max(h(\alpha_j), |\log \alpha_j|/D, 1/D)$ for $j = 1,2$. Let $\Lambda = b_1 \log \alpha_1 - b_2 \log \alpha_2$, $b_1, b_2 \in \boldsymbol{N}$. Then we have

$$\log |\Lambda| \geq -32.31\, D^4 (\log A_1)(\log A_2) \cdot \Big(\max\Big(\frac{10}{D}, 0.18 + \log B\Big)\Big)^2,$$

where $B = b_1/D \log A_2 + b_2/D \log A_1$.

Proof. Letting $h_2 = 10$ in the Table 2 in [3], we obtain this lemma immediately in the same way as Corollary 2 of [3].

3. Proof of theorem. We now assume that r and s satisfy $2 \| r$, $s \equiv 3 \pmod 4$ and $r \geq 81s$. Let (x, y, z) be a solution of (3) with $(x, y, z) \neq (2,2,2)$. Then, by Lemma 2, we have

(4) $$a^x + b = c^z,\ 2 \mid x,\ 2 \nmid z.$$

Further, by the proof of [1, Theorem], we have $z < x$.

Since $c = a + 2s^2$, we get from (2) that

(5) $$\log c = \log a + \rho_1,$$

where ρ_1 satisfies

(6) $$0 < \rho_1 = \frac{2s^2}{r^2} \sum_{k=0}^{\infty} \frac{1}{2k+1} \Big(\frac{s^2}{r^2}\Big)^{2k}$$

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$$\leq \frac{2}{81^2}\sum_{k=0}^{\infty}\frac{81^{-4k}}{2k+1} < 0.0003049.$$

Similarly, we get from (4) that

(7) $$z\log c - x\log a := \rho_2,$$

where ρ_2 satisfies

(8) $$0 < \rho_2 = \frac{2b}{a^x + c^z}\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\frac{b}{a^x + c^z}\right)^{2k} < \frac{b}{a^x}.$$

The combination of (5), (6) and (7) yields

(9) $$z = \frac{(x-z)\log a + \rho_2}{\rho_1} > \frac{\log a}{\rho_1} > 3279\log a.$$

Let $B = z/\log a + x/\log c$. Then we have

(10) $$B = \frac{2x}{\log c} + \frac{\rho_2}{(\log a)(\log c)}.$$

By Lemma 3, if $B \leq e^{9.82}$, then we get

(11) $$\log\rho_2 \geq -3231(\log a)(\log c).$$

From (8) and (11), we obtain

(12) $$\frac{\log b}{\log a} + 3231\log c > x.$$

The combination of (9) and (12) yields

$$1 + 3231\log c > x > z > 3279\log a = 3279\log c - 3279\rho_1 > 3279\log c - 1.1,$$

a contradiction.

On the other hand, by Lemma 3, if $B > e^{9.82}$, then

(13) $$\log\rho_2 \geq -32.31(\log a)(\log c)(0.18 + \log B)^2.$$

From (8) and (13), we get

$$1 + 64.62(0.18 + \log B)^2 > \rho_2 + \frac{2\log b}{(\log a)(\log c)} + 64.62(0.18 + \log B)^2 > B,$$

whence we conclude that $B > 4860$, a contradiction. Thus, the theorem is proved.

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