On the Solvability of Linear Partial Differential Equations

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1. Introduction. In this note, we introduce a purely algebraic method to prove the solvability of partial differential equations. In particular, our techniques enable us to treat $\mathcal{D}_X$-modules which are not hyperbolic nor simple characteristic.

2. Basic ideas to prove the solvability for $\mathcal{D}_X$-modules. Let M be a real analytic manifold and $N = \{x_1 = 0\}$ its closed submanifold of codimension 1. Denote by $Y \subset X$ a complexification of $N \subset M$. Let $\mathcal{M}$ be a coherent module over the sheaf of ring $\mathcal{D}_X$ of holomorphic differential operators on X and we always assume that Y is noncharacteristic for $\mathcal{M}$. We also denote by $\mathcal{B}_M$ (resp. $\mathcal{C}_M$) the sheaf of Sato's hyperfunctions (resp. microfunctions) on M. The following theorem is a natural generalization of a theorem on the solvability of single differential equations (Theorem 6.5 of Kashiwara-Kawai [6]) to the systems of differential equations, that is, to $\mathcal{D}_X$-modules.

Theorem 2.1. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module for which Y is noncharacterisitic. Assume that $\mathcal{M}$ is micro-hyperbolic (in the sense of Kashiwara-Kawai [6] or Kashiwara-Schapira [7]) in the direction $\pm dx_1 \in \dot{T}^*_N M$ on $N \times_M \dot{T}^*_M X$. Then we have the vanishing of cohomologies:*

$$\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)\,|_N \simeq 0 \text{ for } j > \operatorname{proj.dim}\mathcal{M}_Y,$$

where $\mathcal{M}_Y$ is the induced system of $\mathcal{M}$ on Y defined in [14].

Proof. First of all, there is a distinguished triangle:

$$RHom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)\,|_M \to RHom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M) \to R\dot{\pi}_* RHom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_M) \to +1,$$

where $\dot{\pi} : \dot{T}^*_M X \to M$ is the projection. Since we know by the Cauchy-Kowalevski theorem $\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)\,|_N \simeq 0$ for $j > \operatorname{proj.dim}\mathcal{M}_Y$, it is enough to vanish the complex for open subsets $U \subset N$:

$$R\Gamma(U ; R\dot{\pi}_* RHom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_M)\,|_N) \simeq R\Gamma(U \times_M \dot{T}^*_M X ; RHom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_M)).$$

Thanks to the micro-hyperbolicity and the division theorem of Kashiwara-Kawai [5], this complex is a direct summand of the complex $R\Gamma(U \times_N \dot{T}^*_N Y ; RHom_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N))$. Hence the assertion follows from the "flabbiness" of the sheaf $\mathcal{C}_N$ of the microfunctions on the initial hypersurface N. ■

Remark 2.2. The condition of Kashiwara-Kawai [6], so-called partial micro-hyperbolicity is weaker than that of ours. They treat only single equations but they do not assume the micro-hyperbolicity in both sides. Now it is only an exercise to recover their theorem by using the complex $\mathcal{C}_{\Omega|X}$ of Schapira [15] and the techniques of the proof above.

Remark 2.3. The methods in the proof of Theorem 2.1 was first introduced in the master thesis [11] of the first author H.K., being suggested by the second author K.T. It would be possible to obtain also the tempered version (that is in the space of Schwartz's distribution) of Theorem 2.1 if we use a very recent result of F. Tonin [19] and the suppleness of the sheaf of tempered microfunction proved by Bengel-Schapira [1].

3. Some variants for single differential equations. Recall that the hypersurface N is defined by $\{x_1 = 0\}$ in M and we set $\Omega := \{x_1 > 0\} \subset M$. We shall use the complex $\mathcal{C}_{\Omega|X}$ of sheaves on T^*X introduced by Schapira [15] to obtain the half version of Theorem 2.1.

First let us take a coordinate system $(z ; \zeta dz)$, $z = x + \sqrt{-1}\,y$, $\zeta = \xi + \sqrt{-1}\,\eta$ of T^*X and the associated coordinate system

$$\left((x + \sqrt{-1}y ; \xi + \sqrt{-1}\eta),\ x^* \frac{\partial}{\partial x} + y^* \frac{\partial}{\partial y} + \xi^* \frac{\partial}{\partial \xi} + \eta^* \frac{\partial}{\partial \eta}\right)$$

of the tangent bundle $T((T^*X)^{\mathbf{R}})$ of the underlying real analytic manifold $(T^*X)^{\mathbf{R}}$ of T^*X. Next take a point $p := (x ; \sqrt{-1}\,\eta)$ of $N \times_M$

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T_M^*X and identify T_x^*M with a closed subspace of $T_p((T^*X)^{\boldsymbol{R}})$ via the Hamiltonian isomorphism as in [17]:

$$-H: T_x^*M \ni (x; x^*dx) \mapsto \left((x; \sqrt{-1}\eta); x^*\frac{\partial}{\partial\xi}\right) \in T_p((T^*X)^{\boldsymbol{R}}).$$

In this situation, we are announced the theorem below by Uchida [20], which is a consequence of Corollary 6.4.3 of Kashiwara-Schapira [8].

Theorem 3.1. (*Uchida* [20]) *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module on X and $p = (x; \sqrt{-1}\,\eta) \in N \times_M T_M^*X$. Assume $\mathcal{M}$ satisfies the condition*:

$$-H(\theta) \notin C_p(Ch\mathcal{M}, \bar{\Omega} \times_M T_M^*X)$$

for $\theta = (x + dx_1) \in T_M^*X$. Then we have: $RHom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{M_+|X}) \simeq 0$ at p, where $\mathcal{C}_{M_+|X}$ is the sheaf introduced by Kataoka [9].

Now we can easily obtain the following result by using the techniques in the proof of Theorem 2.1 and Uchida's theorem above if we consider the following distinguished triangle by setting $\dot{\pi}_X: \dot{T}^*X \to X$:

$$RHom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_{\bar{\Omega}} \to RHom_{\mathcal{D}_X}(\mathcal{M}, \Gamma_\Omega\mathcal{B}_M) \to R\dot{\pi}_{X*}RHom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{\Omega|X}) \to +1.$$

Theorem 3.2. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module for which Y is noncharacteristic. Assume that $\mathcal{M}$ satisfies the condition*:

$$-H(\theta) \notin C_p(Ch\mathcal{M}, \bar{\Omega} \times_M T_M^*X)$$

for any $p = (x; \sqrt{-1}\,\eta) \in N \times_M \dot{T}_M^*X$ and $\theta = (x; +dx_1) \in T_x^*M$. Then we have:

$$\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \Gamma_\Omega\mathcal{B}_M)\,|_N \simeq 0 \textit{ for } j > \text{proj.dim}\mathcal{M}_Y.$$

Example 3.3. Let $P = EQ +$ (lower order terms) be a differential operator on X with E elliptic and Q semi-hyperbolic in $+dx_1$ direction in the sense of Schapira-Zampieri [17]. Then

$$P: \Gamma_\Omega\mathcal{B}_M\,|_N \to \Gamma_\Omega\mathcal{B}_M\,|_N$$

is surjective. For example take $Q = D_1^2 - x_1^k D'^2$ for $k \geq 2$. Kataoka's result on the solvability of semi-hyperbolic pseudo-differential operators (Corollary 1.9 of [9]) enables us to take as Q any semi-hyperbolic operators in the sense of Kaneko [4]. In particular, we can treat also the case when $k = 1$, that is, when Q is the Tricomi operator. We thank Kataoka for the useful discussions on this point.

Remark 3.4. On the other hand, Oaku [13] has obtained a general theorem on the solvability of the homogeneous boundary value problems for single differential equations. Thus the example above means that we can treat also inhomogeneous problems by using mild hyperfunctions.

Finally we shall treat a case which cannot be treated by Theorem 6.5 of Kashiwara-Kawai [6]. We assume $M = M' \times M'' = \boldsymbol{R}^d \times \boldsymbol{R}^{n-d}$, $X = X' \times X'' = \boldsymbol{C}^d \times \boldsymbol{C}^{n-d}$ and $N = \{x_1 = 0\} = \boldsymbol{R}^{d-1} \times \boldsymbol{R}^{n-d} \subset M$. The following result is already announced in Koshimizu [11]. The details will appear in a subsequent paper [12].

Theorem 3.5. *Let E, $Q \in \mathcal{D}_{X'}$ be differential operators on X'. Assume that E is elliptic and Q is hyperbolic in $\pm\, dx_1$-direction We set $P := EQ +$ (lower order terms) $\in \mathcal{D}_X$ by taking arbitary lower order terms from the differential operators on the total space $X \simeq \boldsymbol{C}^n$. Then we have*:

(i) *$P: \mathcal{B}_M \to \mathcal{B}_M$ is surjective.*

(ii) *$P: \mathcal{C}_M \to \mathcal{C}_M$ is surjective at any point in $\dot{T}_M^*X$.*

To prove this theorem we made use of the theory of bimicrolocalization introduced in [16] and [18] and the techniques in the proof of Theorem 2.1. We also require the flabbiness of the sheaf of second microfunctions proved in Kataoka-Tose [10].

Remark 3.6. There are some general theories on the solvability of differential operators with multiple characteristic (that is, with singular points in the characteristic variety on $\dot{T}_M^*X$), but it is a difficult problem in general. One of the advantage of employing the hyperfunction theory is that we can sometimes prove the solvability even for the operators with multiple characteristic. For example, Kashiwara-Kawai [6] and Kashiwara-Schapira [7] treat micro-hyperbolic operators in the sheaf $\mathcal{C}_M$ of microfunctions, and Bony-Schapira [2] proved the solvability of partially elliptic equations. The operators treated in Theorem 3.5 are not micro-hyperbolic nor partially elliptic along a part of their characteristic varieties. Therefore the solvability does not follow from general theories. We have to blow up along the singular points of the characteristic variety and consider the problem second microlocally. The methods of Delort [3] was also useful to prove Theorem 3.5.

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