The Maximal Finite Subgroup in the Mapping Class Group of Genus 5

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Abstract: The automorphism groups of compact Riemann surfaces of genus 5 are enumerated by A. Kuribayashi and H. Kimura. Among them, the group of largest order is a group of order 192. The Riemann surface with this automorphism group is unique, and it is realized as the modular curve $X(8)$ of level 8. By utilizing this, we have explicit construction of the finite subgroup of order 192 in the Teichmüller group of genus 5.

0. Introduction. The compactified modular curve $X(8)$ of level 8 corresponding to the principal congruence subgroup $\Gamma(8)$ of $\Gamma(1) = SL_2(\boldsymbol{Z})$ defines a compact complex algebraic curve of genus 5. We are interested in the following problem. Its modulus $[X(8)]$ in the moduli space $\mathcal{M}_5$ of genus 5 curves defines a (singular) point. $\mathcal{M}_5$ is given as a quotient space $\Gamma_5 \backslash \mathcal{T}_5$ of the Teichmüller space $\mathcal{T}_5$ of genus 5 by the Teichmüller group Γ_5 of genus 5. Let $[X(8)]^{\sim}$ be a point of $\mathcal{T}_5$ corresponding to a marking $\beta : \pi_1(X(8), *) \simeq \pi_5$, here π_5 is the surface group of genus 5. Then by a Theorem of Kerckhoff ([1]), the stabilizer of $[X(8)]^{\sim}$ in Γ_5 is isomorphic to the automorphism group $\mathrm{Aut}(X(8)) \cong SL_2(\boldsymbol{Z}/8\boldsymbol{Z})/\{\pm 1\}$. Our problem is to give an explicit description of this stabilizer in $\Gamma_5 = \mathrm{Out}^+(\pi_5)$ in terms of canonical basis of π_5. The same problem for the Klein curve $X(7)$ of genus 3 have been solved by Matsuura using different ideas. ([5])

1. Some general facts. First we briefly describe the well-known construction of the canonical generators in the fundamental group of compact Riemann surface $X_\Gamma = \Gamma \backslash \mathfrak{H}^*$ corresponding to a Fuchsian group of first kind $\Gamma \subset SL_2(\boldsymbol{R})$ ([3]). We are interested in the case when the action of Γ on $\mathfrak{H}$ is fixed-point free. Choose a base point $\Gamma x_0 \in X_\Gamma$, take as a fundamental domain of X_Γ the domain

$$\mathfrak{D} = \bigcap_{\gamma \in \Gamma} \{x \in \mathfrak{H} \mid d(x, x_0) \leq d(x, \gamma x_0)\},$$

where d is $SL_2(\boldsymbol{R})$-invariant metric on $\mathfrak{H}$. Choose an orientation from left to right on the boundary of $\mathfrak{D}$. Each side a of $\mathfrak{D}$ has its conjugate a^{-1}, let $\gamma_a \in \Gamma$ be a map $a \to a^{-1}$. Denote by $\delta(a)$, the homotopy class of the loop $\delta_1\delta_2$, where δ_1 is a path from x_0 to the endpoint of a and δ_2 is a bath from initial point of a^{-1} to x_0. Then for any relation $\Pi\, a_i^{\pm 1} = 1$ among boundary sides we have $\Pi\, \delta(a_i^{\pm 1}) = 1$ with the same exponents. Thus, we have $\delta(a^{-1}) = \delta(a)^{-1}$. There is another important relation between our loops: for a vertex P of $\mathfrak{D}$ let $a(P)$ be the boundary side starting at P, denote $\sigma(P) = \gamma_{a(P)}(P)$. The cycle of vertex P is a finite set of vertices $\{\sigma^n(P) \mid n \in \boldsymbol{N}\}$. When the cycle of P is $\{P, \sigma(P), \ldots, \sigma^k(P)\}$, we have a relation $\Pi_{i=0}^k \delta(a(\sigma^i(P))) = 1$. After eliminating these relations from the fundamental relation, we will get a relation in exactly $2g$ loops, which generate the fundamental group $\pi_1(X_\Gamma, \Gamma x_0)$, here $g = \mathrm{genus}(X_\Gamma)$.

Suppose that, in the fundamental relation two sides a, b and their conjugates a^{-1}, b^{-1} occur in the order $\ldots a \ldots b \ldots a^{-1} \ldots b^{-1} \ldots$. That is , we can write the fundamental relation as $aWbXa^{-1}Yb^{-1}Z = 1$, where W, X, Y, Z are blocks of sides. Firstly, we denote $e = WbX$, our relation transforms to $aea^{-1}YXe^{-1}WZ = 1$ (gluing b on b^{-1}), secondly denote $d = X^{-1}Y^{-1}a$ then, we get a relation $ded^{-1}e^{-1}WZYX = 1$ (gluing a on a^{-1}). After g times repetitions of this procedure we find a generator system with relation $\Pi_{i=1}^g [d_i, e_i] = 1$, here $[a, b]$ is the commutator $aba^{-1}b^{-1}$. ([3] section 7.4)

Let now $x_1, x_2 \in \mathfrak{H}^*$ be two points such that $\Gamma x_1 = \Gamma x_2 = \Gamma x_0$. Then the path δ connecting x_1, x_2 in $\mathfrak{H}^*$ defines a closed path on $X_\Gamma(\mathbf{C})$, therefore its homotopy class in $\pi_1(X_\Gamma, \Gamma x_0)$ can be expressed in terms of our canonical generators. The following simple argument give us one such expression. Assume that, δ intersects with the boundary side a of $\mathscr{D}$ and a terminal point x of δ is lying outside of $\mathscr{D}$. Denote the arc of δ lying outside of $\mathscr{D}$ by ν. Then by the transformation map γ_a, ν goes either to an union of arcs ν_1, ν_2, where ν_1 is lying inside of $\mathscr{D}$ while ν_2 is not, or to a single arc lying entirely in $\mathscr{D}$. In the first case of course we have $d(\nu_2) < d(\nu)$, so after finite number of repetition of these procedure we will get the arcs all lying in $\mathscr{D}$, the union of which is Γ-equivalent to ν. Then homotopy class of δ is easily determined by the boundary sides of $\mathscr{D}$, and next we can express it in terms of canonical generators.

2. Generators of the fundamental group of $X(8)$. Now we treat the main result of this paper, the action of the automorphism group of the modular curve $X(8) = X_{\Gamma(8)}$ on its fundamental group. The curve $X(8)$ has a particular interest, because it has largest automorphism group among the curves of genus 5.([2])

As to the fundamental domain of $\Gamma(8)$ acting on the upper half-plane $\mathfrak{H}$, we can take a domain

$$\mathscr{D} = \left\{z \in \mathfrak{H} \mid 0 \leq |Re(z)| \leq 8, |cz+d| \geq 1 \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(8)\right\}.$$

By adding the cusps $\Gamma(8) \backslash \boldsymbol{P}^1_{\boldsymbol{Q}}$ to $\mathscr{D}$ and gluing the boundaries we obtain the modular curve $X(8)$ of genus 5. The boundaries of $\mathscr{D}$ are identified as:

$$[\infty, 0] \leftrightarrow [\infty, 8],\ \left[k, \frac{1}{4}+k\right] \leftrightarrow \left[k, -\frac{1}{4}+k\right],$$
$$\left[\frac{1}{4}+k, \frac{1}{3}+k\right] \leftrightarrow \left[-\frac{9}{4}+k, -\frac{7}{3}+k\right],$$
$$\left[\frac{1}{3}+k, \frac{3}{8}+k\right] \leftrightarrow \left[-\frac{7}{3}+k, -\frac{19}{8}+k\right],$$
$$\left[\frac{3}{8}+k, \frac{1}{2}+k\right] \leftrightarrow \left[-\frac{27}{8}+k, -\frac{7}{2}+k\right],$$

here $k = 0, \ldots, 7$ and for the points P, Q on the real axis, $[P, Q]$ is the oriented semi-circle from P to Q in $\mathfrak{H}$ centered on the real axis.

In general for any subgroup Γ of $\Gamma(1)$ of finite index acting freely on $\mathfrak{H}$, the fundamental group $\pi_1(X_\Gamma, *)$ is generated by modular symbols, that are loops connecting cusp points. ([4] Prop.1.4) Choose as a base point x_0 the cusp $\Gamma(8)\,\frac{3}{8}$, then the fundamental group $\pi_1(X(8), x_0)$ is generated by the homotopy classes of sixteen semi circles

$$a_i = \left[\frac{3}{8}+i, \frac{5}{8}+i\right],\ b_j = \left[\frac{5}{8}+2j, \frac{11}{8}+2j\right],\ c_j = \left[\frac{13}{8}+2j, \frac{19}{8}+2j\right];$$

$i = 1, \ldots, 8$, $j = 1, \ldots, 4$. The cycles of vertices in $\mathscr{D}$ will give us $a_{i+4} = a_i^{-1}$; $i = 1, \ldots, 4$ and $b_4 = b_1^{-1}b_2^{-1}b_3^{-1}$, $c_4 = c_1^{-1}c_2^{-1}c_3^{-1}$. Hence the fundamental relation in $\pi_1(X(8), x_0)$ reads now as $a_1b_1a_2c_1a_3b_2a_4c_2a_1^{-1}b_3a_2^{-1}c_3a_3^{-1}b_1^{-1}b_2^{-1}b_3^{-1}a_4^{-1}c_1^{-1}c_2^{-1}c_3^{-1} = 1$.

Let $\delta \in \{a_i, b_j, c_j \mid i = 1, \ldots, 4, j = 1, \ldots, 3\}$ and $\gamma \in \Gamma(1)$. By definition $\gamma \cdot \delta$ is the homotopy class of $t_\gamma[x_1, x_2]t_\gamma^{-1}$, here t_γ denotes a path from $\frac{3}{8}$ to $\gamma\left(\frac{3}{8}\right)$ fixed once for all and $x_1 = \gamma(u_1)$, $x_2 = \gamma(u_2)$ if δ is the homotopy class of $[u_1, u_2]$. We need only to find actions of the matrices $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of $\Gamma(1)$ on our generators of the fundamental group, since they generate $\Gamma(1)$. Choose as t_T zero-path from $\frac{3}{8}$ to $\frac{3}{8}$, and $t_J = \left[-\frac{21}{8}, -\frac{8}{3}\right] = \left[\frac{21}{8}, \frac{8}{3}\right]$.

Proposition 1. *Let a_i, b_j, c_j; $i = 1, \ldots, 4$, $j = 1, \ldots, 3$ are the generators of the fundamental group and the matrices $T, J \in \Gamma(1)$ are as above. We have*

$$\begin{aligned}
T\cdot a_1 &= a_2, & J\cdot a_1 &= a_2^{-1}c_3\\
T\cdot a_2 &= a_3, & J\cdot a_2 &= b_3^{-1}\\
T\cdot a_3 &= a_4, & J\cdot a_3 &= b_2\\
T\cdot a_4 &= a_1^{-1}, & J\cdot a_4 &= a_3^{-1}c_1^{-1}\\
T\cdot b_1 &= c_1, & J\cdot b_1 &= a_3^{-1}b_1^{-1}b_2^{-1}\\
T\cdot b_2 &= c_2, & J\cdot b_2 &= a_3\\
T\cdot b_3 &= c_3, & J\cdot b_3 &= a_2^{-1}\\
T\cdot c_1 &= b_2, & J\cdot c_1 &= a_4^{-1}b_2^{-1}\\
T\cdot c_2 &= b_3, & J\cdot c_2 &= c_2^{-1}\\
T\cdot c_3 &= b_1^{-1}b_2^{-1}b_3^{-1}, & J\cdot c_3 &= b_3^{-1}a_1.
\end{aligned}$$

Proof. Here we explain only the first row. It is obvious that, $T\cdot a_1 = a_2$ and $J\cdot a_1 = \left[-\frac{21}{8}, -\frac{8}{3}\right]\left[-\frac{8}{3}, -\frac{8}{5}\right]\left[-\frac{8}{3}, -\frac{21}{8}\right] = \left[-\frac{21}{8}, -\frac{13}{8}\right]\left[-\frac{13}{8}, -\frac{8}{5}\right]\left[\frac{8}{3}, \frac{21}{8}\right]$. By the transformation matrix $\begin{pmatrix} 41 & 64 \\ 16 & 25 \end{pmatrix} : \left[-\frac{13}{8}, -\frac{3}{2}\right] \to \left[\frac{5}{2}, \frac{21}{8}\right]$ the semi-circle $\left[-\frac{13}{8}, -\frac{8}{5}\right]$ is equivalent to $\left[\frac{21}{8}, \frac{8}{3}\right]$. Thus we get $J\cdot a_1 = \left[-\frac{21}{8}, -\frac{13}{8}\right]$ or in our notation $a_2^{-1}c_3$. The action to other generators are obtained by same method.

We construct canonical generators using the procedure from the preceding section. Namely, by gluing the edges in the following order b_3, c_3 ; c_1, b_2 ; b_1, c_2^{-1} ; a_2^{-1}, a_1 ; a_4^{-1}, a_3, we obtain following result.

Proposition 2. *Homotopy classes of the following set of loops can be taken as a canonical generator system of the fundamental group* $\pi_1\left(X(8), \Gamma(8)\frac{3}{8}\right)$:

$$\begin{aligned} d_1 &= [e_5, d_5]a_4^{-1}, & e_1 &= [d_2, e_2]a_3[d_5, e_5], \\ d_2 &= a_4a_1c_2^{-1}a_4^{-1}b_1a_3c_2c_1, & e_2 &= a_3b_2a_4c_2a_1^{-1}a_4^{-1}, \\ d_3 &= a_3a_4^{-1}b_1, & e_3 &= a_2c_2^{-1}a_3^{-1}, \\ d_4 &= [e_3, d_3]a_3a_4^{-1}a_2^{-1}, & e_4 &= a_1a_4a_3^{-1}[d_3, e_3], \\ d_5 &= b_2b_1a_3c_2c_1a_4b_3, & e_5 &= a_2^{-1}c_3a_3^{-1}b_1^{-1}b_2^{-1}, \end{aligned}$$

They satisfy a relation $\Pi_{i=1}^{5}[d_i, e_i] = 1$.

Note that, reverse to these substitution one has:

$$\begin{aligned} a_1 &= e_4[e_3, d_3][e_2, d_2]e_1d_1, \\ a_2 &= d_4^{-1}[e_3, d_3][e_2, d_2]e_1d_1, \\ a_3 &= [e_2, d_2]e_1[e_5, d_5], \\ a_4 &= d_1^{-1}[e_5, d_5], \\ b_1 &= d_1^{-1}e_1^{-1}[d_2, e_2]d_3, \\ b_2 &= [d_5, e_5]d_1e_2^{-1}[d_2, e_2]e_2d_1^{-1}[e_5, d_5]e_4d_4e_3[e_2, d_2] \\ &\quad \times e_1d_1, \\ b_3 &= [d_5, e_5]d_1e_2^{-1}d_2^{-1}e_1[e_5, d_5]d_5, \\ c_1 &= d_1^{-1}e_1^{-1}[d_2, e_2][d_3, e_3]d_4d_3^{-1}[d_3, e_3]d_4^{-1} \\ &\quad \times e_4^{-1}[d_5, e_5]d_1d_2, \\ c_2 &= [d_5, e_5]e_1^{-1}[d_2, e_2]e_3^{-1}d_4^{-1}[e_3, d_3][e_2, d_2]e_1d_1, \\ c_3 &= d_4^{-1}[e_3, d_3][e_2, d_2]e_1d_1e_5[d_5, e_5]e_1^{-1}[d_2, e_2]e_2 \\ &\quad \times d_1^{-1}[e_5, d_5]e_4d_4e_3d_3[e_2, d_2]e_1[e_5, d_5]. \end{aligned}$$

Using this formula and Proposition 1-2 we obtain a formula for the action of automorphism group on our canonical generators of $\pi_1(X(8), *)$.

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References

[1] S. P. Kerckhoff: The Nielsen realization problem. Ann. Math., **117**, 235–265 (1983).

[2] A. Kuribayashi and H. Kimura: Automorphism groups of compact Riemann surfaces of genus five. J. Algebra, **134**, 80–103 (1990).

[3] J. Lehner: Discontinuous Groups and Automorphic Functions. American Mathematical Society, Providence, Rhode Island (1964).

[4] Yu. I. Manin: Parabolic points and zeta-functions of modular curves. Math. USSR. Izvestiya, **36**, 19–64 (1972).

[5] A. Matsuura: The automorphism group of the Klein curve in the mapping class group of genus 3 (preprint).