An Extension of Sturm's Theorem to Two Dimensions

By Tomokatsu SAITO

Department of Mathematics, Sophia University

(Communicated by Shokichi IYANAGA, M. J. A., Jan. 13, 1997)

1. Introduction and notation. Let $f(x, y) \in \boldsymbol{R}[x, y]$ be a square free polynomial with real coefficients, namely $f(x, y)$ is decomposed into the irreducible factors whose multiplicities are only one. Let C be the set of points $(x, y) \in \boldsymbol{R}^2$ such that $f(x, y) = 0$. Until now, only the following primitive method has been used to draw the curve C by computer, within a given rectangle R. We decompose R into many small rectangles D and obtain $C \cap R$ by gathering $C \cap D$. $C \cap D$ is found as follows.

Let D be the set $\{(x, y) \in \boldsymbol{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$, and put $P_1 = (a, c)$, $P_2 = (b, c)$, $P_3 = (b, d)$ and $P_4 = (a, d)$. For example, if $f(P_1)f(P_2) < 0$, $f(P_3)f(P_4) < 0$ then we can find approximately a point P_5 in $C \cap \overline{P_1P_2}$ and a point P_6 in $C \cap \overline{P_3P_4}$. Then the line $\overline{P_5P_6}$ can be considered approximately as $C \cap D$.

But the above method has next two problems.

(1) Even if $f(P_1)f(P_2) > 0$, it is possible that $C \cap \overline{P_1P_2} \neq \emptyset$.

(2) Even if $C \cap$ (the boundary of D) $= \emptyset$, it is possible that $C \cap$ (the interior of D) $\neq \emptyset$.

In this paper, we would like to propose a more reliable method which permit us to liberate from these incertainties.

Let ∂D be the boundary of D and D^i be the interior of D. Then $C \cap D$ is the direct union of $C \cap \partial D$ and $C \cap D^i$. The search for $C \cap D$ is made separately in two cases: the first case for $C \cap \partial D$ and the second case for $C \cap D^i$.

2. First case. This case can be treated as the equation $f = 0$ is restricted to a boundary line. Then we can use Sturm's theorem.

The Sturm sequence associated with the (one-variable) polynomial $f(x)$ is a sequence of polynomials with $f_0(x), f_1(x), \ldots, f_k(x)$ defined by the following equations:

$$f_0(x) = f(x), f_1(x) = f'(x),$$
$$f_i(x) = -\text{ remainder } (f_{i-2}(x), f_{i-1}(x))$$

where remainder means the remainder from the division of the former by the latter.

Let $(a_1, \ldots, a_s)$ be a sequence of real numbers and $(a'_1, \ldots, a'_t)$ be the subsequence of all non-zero numbers. Then $var(a_1, \ldots, a_s)$, the number of sign variations, is the number of i, $1 \leq i < t$, such that $a'_i a_{i+1'} < 0$.

Theorem (Sturm). *Let $f(x)$ be a square free polynomial. When $gcd(f(x), f'(x)) = f_k(x)$, the number of real roots of $f(x)$ in the interval $a < x \leq b$ is*

$$var(f_0(a), f_1(a), \cdots, f_k(a)) -$$
$$var(f_0(b), f_1(b), \cdots, f_k(b)).$$

Let D be the set $\{(x, y) \in \boldsymbol{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$. Using Sturm's theorem we can determine whether $f(x, c) = 0$ has a root in the interval $[a, b]$ of not. Thus we can determine whether $C \cap \partial D \neq \emptyset$ or not, and if $C \cap \partial D \neq \emptyset$, find this set approximately in considering from divisions of ∂D.

3. Second case. When $C \cap \partial D = \emptyset$ then we can find $C \cap D^i$ in the following manner.

If $C \cap D^i \neq \emptyset$, then there is a point (x_0, y_0) such that $(x_0, y_0) \in C \cap D^i$, but if $(x, y) \in C \cap D^i$, then $y \leq y_0$. Such a point (x_0, y_0) will be called a *maximal point* (of $C \cap D^i$ with respect to y). We write $f_x(x, y) = \dfrac{\partial}{\partial x} f(x, y)$ and show $f_x(x_0, y_0) = 0$ for a maximal point (x_0, y_0). If $f_x(x_0, y_0) \neq 0$ then using implicit function theorem, there exists a function $g(y)$ near y_0 such that $f(g(y), y) = 0$ and (x_0, y_0) cannot be a maximal point. Therefore we have $f_x(x_0, y_0) = 0$.

As $f(x, y)$ is square free, we have $gcd(f(x, y), f_x(x, y)) = 1$ in $\boldsymbol{R}(y)[x]$. Using Euclidean algorithm we can find $g(x, y), h(x, y) \in \boldsymbol{R}[x, y]$, $F(y) \in \boldsymbol{R}[y]$ such that

(3) $f(x, y)g(x, y) + f_x(x, y)h(x, y) = F(y)$

If $f(x, y) = 0$, $f_x(x, y) = 0$, then $F(y)$ must be zero. Using Sturm's theorem, we can count correctly the number of roots $F(y) = 0$ in the interval $[c, d]$ and we can calculate approximately all roots in this interval. Therefore we can calculate all points (x, y) such that $f(x, y) = 0$ and $f_x(x, y) = 0$. Thus we can decide whether $C \cap D^i \neq \emptyset$ or not. Even if the set $C \cap D^i$ is only one point, we can find the point by this method.

4. Determination of whether $C = \emptyset$ or not. Let $D_1 = \{(x, y) \mid |x| = 1 \text{ or } |y| = 1\}$ and $D_2 = \{(x, y) \mid |x| < 1 \text{ or } |y| < 1\}$. Using Sturm's theorem we can determine whether $D_1 \cap C = \emptyset$ or not. When $D_1 \cap C = \emptyset$ we can determine whether $D_2 \cap C = \emptyset$ or not by the above method. Let $D_3 = \{(x, y) \mid |x| > 1, |y| > 1\}$. We can determine whether $D_3 \cap C = \emptyset$ or not as follows.

When $f(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{i,j} x^i y^j$ where for some i, $a_{i,n} \neq 0$ and for some j, $a_{m,j} \neq 0$, we put $g(x, y) = x^m y^n f(1/x, 1/y)$. Let $C' = \{(x, y) \mid g(x, y) = 0\}$, $D_4 = \{(x, y) \mid |x| < 1, |y| < 1\}$, $D_5 = \{(x, y) \in D_4 \mid x = 0 \text{ or } y = 0\}$. $D_5 \cap C'$ is a finite set of points $\{p_1, p_2, \ldots, p_s\}$ which can be computed by Sturm's theorem. Let D'_i be a small rectangle such that $p_i \in D'_i \subset D_4$, $p_i \notin D'_j (i \neq j)$. We can determine whether $D'_i \cap C' = \{p_i\}$ or not again by the above method. Therefore we can determine $D_3 \cap C = \emptyset$ or not.

5. Use of Gröbner basis. Let $I = < f(x, y), f_x(x, y) >$ be the ideal in $\boldsymbol{R}[x, y]$ generated by $f(x, y)$ and $f_x(x, y)$. Using the Gröbner algorithm we can find a Gröbner basis of I. A Gröbner basis of I is a basis of I which has the next desirable property. If $I \cap \boldsymbol{R}[y] = \langle F_0(y) \rangle$, then $F_0(y)$ is a member of Gröbner basis (Lemma 6.50 [3]). $F_0(y)$ is a devisor of $F(y)$ in (3). Occasionally the degree of $F(y)$ becomes very large even if the degree of $F_0(y)$ is small. As the set $S = \{y \mid f(x, y) = 0, f_x(x, y) = 0 \text{ for some } x\}$ is finite, we put $S = \{y_1, \ldots, y_n\}$. From (3) we have $F_0(y_i) = 0$. Let $G(y)$ be $(y - y_1)(y - y_2) \cdots (y - y_n)$. From Hilbert Nulstellensatz, some power of $G(y)$ must be in I. Therefore if $F_0(y_0) = 0$ then $y_0 = y_i$ for some $i (1 \leq i \leq n)$. As we have an algorithm to calculate $F_0(y)$, (cf. [3]), it is more advantegeous to use it.

References

[1] G. E. Collins and R. Loos: Real zeros of polynomials. Computer Algebra Symbolic and Algebraic Computation (eds. B. Buchberger, G.E. Collins, and R. Loos). Springer-Verlag, New York, pp. 83–94 (1983).

[2] D. E. Knuth: The Art of Computer Programming. second ed., vol. 2, Addison-Wesley (1981).

[3] T. Becker and V. Weispfenning: Gröbner Basis. Springer-Verlag, New York (1993).