A Theta Product Formula for Jackson Integrals Associated with Root Systems

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Jackson integrals associated with root systems. Let $\mathfrak{a}$ be an n-dimensional vector space over $\boldsymbol{R}$ with an inner product $\langle\cdot,\cdot\rangle$. Let $R\subset\mathfrak{a}^*$ be an irreducible reduced root system and W_R be the group generated by orthogonal reflections with respect to the hyperplane perpendicular to $\alpha\in R$, the so-called *Weyl group* associated with R. Let P be the *weight lattice* of R defined by $\{\mu\in\mathfrak{a}^*;\langle\mu,\alpha^\vee\rangle\in\boldsymbol{Z}$ for any $\alpha\in R\}$, where $\alpha^\vee=2\alpha/\langle\alpha,\alpha\rangle$. We fix a base $\{\alpha_1,\cdots,\alpha_n\}\subset R$ and its fundamental weights $\{\chi_1,\cdots,\chi_n\}\subset P;\langle\chi_i,\alpha_j^\vee\rangle=\delta_{ij}$. The inner product and the reflections are uniquely extended linearly to $\mathfrak{h}=\boldsymbol{C}\otimes_{\boldsymbol{R}}\mathfrak{a}$. We sometimes identify the vector space $\mathfrak{h}$ with its dual $\mathfrak{h}^*$ via the inner product $\langle\cdot,\cdot\rangle:\mu(\alpha)=\langle\mu,\alpha\rangle$.

Let $\bar{X}$ be an algebraic torus of dimension n, isomorphic to $(\boldsymbol{C}^\times)^n$. We can embed P in $\bar{X}$ by the mapping

$$\mathfrak{h}^*\to\bar{X};\chi=\nu_1\chi_1+\cdots+\nu_n\chi_n \mapsto q^\chi:=(q^{\nu_1},\cdots,q^{\nu_n})$$

where $q=e^{2\pi\sqrt{-1}\tau}$, $\mathrm{Im}\tau>0$. We denote by X the lattice subgroup $\{(q^{\nu_1},\cdots,q^{\nu_n});\nu_i\in\boldsymbol{Z}(i=1,\cdots,n)\}\subset\bar{X}$. We identify P with X. Each $\alpha\in\mathfrak{h}^*$ defines a monomial $t^\alpha:=t_1^{\langle\chi_1,\alpha^\vee\rangle}\cdots t_n^{\langle\chi_n,\alpha^\vee\rangle}$ for $t=(t_1,\cdots,t_n)\in\bar{X}$. To each $\alpha\in R$, let k_α be a complex number such that $k_\alpha=k_\beta$ if $|\alpha|=|\beta|$.

We introduce the following function of $t=(t_1,\cdots,t_n)$ on $\bar{X}$ (see [3]):

$$\Phi_R(k;t)=t^\ell\prod_{\alpha>0}\frac{(q^{1-k_\alpha}t^\alpha)_\infty}{(q^{k_\alpha}t^\alpha)_\infty}$$

where $(x)_\infty=\Pi_{\nu=1}^\infty(1-xq^\nu)$, $\ell=\dfrac{1}{2}\sum_{\alpha>0}(1-2k_\alpha)$ and "$\alpha>0$" means α is a positive root of R. For simplicity we sometimes abbreviate $\Phi_R(k;t)$ by $\Phi_R(t)$. The function $\Phi_R(k;t)$ is quasi-symmetric with respect to W_R:

$$\sigma\Phi_R(k;t)=\Phi_R(k;\sigma^{-1}(t))=U_\sigma(t)\cdot\Phi_R(k;t),\quad\sigma\in W_R$$

where $U_\sigma(t)$ is a *pseudo-constant*, i.e, a q-periodic function with respect to $t\in\bar{X}$ such that

$$U_\sigma(t)=\prod_{\substack{\alpha>0\\ \sigma\alpha<0}}t^{(2k_\alpha-1)\alpha}\frac{\vartheta(q^{k_\alpha}t^\alpha)}{\vartheta(q^{1-k_\alpha}t^\alpha)}$$

for the Jacobi elliptic theta function $\vartheta(x)=(x)_\infty(q/x)_\infty(q)_\infty$. $\{U_\sigma(t)\}_{\sigma\in W_R}$ satisfies the *one cocycle* condition such that $U_{\sigma\sigma'}(t)=U_\sigma(t)\cdot\sigma U_{\sigma'}(t)$.

We let Δ_R denote the *Weyl denominator* defined by $\Delta_R(t):=\Pi_{\alpha>0}(t^{\frac{\alpha}{2}}-t^{-\frac{\alpha}{2}})$. Let us define $\Phi_R'(k;t):=\Phi_R(k;t)\cdot(-1)^{\frac{|R|}{2}}\Delta_R(t)$. Then, the function $\Phi_R'(k;t)$ also has the *quasi-symmetry*

$$\sigma\Phi_R'(k;t)=\mathrm{sgn}\sigma\cdot U_\sigma(t)\cdot\Phi_R'(k;t),\quad\sigma\in W_R.$$

Definition. We now consider the *Jackson integral associated with* R defined by

$$J_R(k;\xi):=\int_{[0,\xi\infty]_q}\Phi_R'(k;t)\frac{d_qt_1}{t_1}\wedge\cdots\wedge\frac{d_qt_n}{t_n} =(1-q)^n\sum_{\chi\in X}\Phi_R'(k;q^\chi\xi)$$

where $\xi=(\xi_1,\cdots,\xi_n)$ is an arbitrary point of $\bar{X}$ and $q^\chi\xi$ means $(q^{\nu_1}\xi_1,\cdots,q^{\nu_n}\xi_n)$.

It is obvious that the Jackson integral $J_R(k:\xi)$ is a q-periodic function of $\xi\in\bar{X}$ if it is convergent:

$$J_R(k;q^\chi\xi)=J_R(k;\xi).$$

Let $\Gamma_q(x)$ denote the q-gamma function $(1-q)^{1-x}(q)_\infty/(q^x)_\infty$.

Conjecture (product formula). *The Jackson integral $J_R(k;\xi)$ can be expressed as follows:*

(1)
$$J_R(k;\xi)=\prod_{\alpha>0}\frac{\Gamma_q(1-\langle\rho_k,\alpha^\vee\rangle)\Gamma_q(-\langle\rho_k,\alpha^\vee\rangle)}{\Gamma_q(1-k_\alpha-\langle\rho_k,\alpha^\vee\rangle)\Gamma_q(k_\alpha-\langle\rho_k,\alpha^\vee\rangle+\delta_\alpha)}\frac{\xi^{-k_\alpha\alpha}(\xi^\alpha)}{\vartheta(q^{k_\alpha}\xi^\alpha)}$$

up to a positive integer, where $\delta_\alpha=1$ if α is a simple root, $\delta_\alpha=0$ otherwise, and $\rho_k=\dfrac{1}{2}\sum_{\alpha>0}k_\alpha\alpha$.

Proposition. *The Jackson integrals of A_n-type, B_2-type and G_2-type have the following formulae:*

$$J_{A_n}(\beta;\xi)=(n+1)\prod_{j=1}^n$$

$$\frac{\Gamma_q(1-\beta)\Gamma_q(1-(n-j+1)\beta)\Gamma_q(-j\beta)}{\Gamma_q(1-j\beta)\Gamma_q(1-(j+1)\beta)}$$

$$\cdot\prod_{i=1}^{n}\xi_i^{(n-2i)\beta}\frac{\vartheta(\xi_i)}{\vartheta(q^\beta\xi_i)}\prod_{0\leq i<j\leq n}\frac{\vartheta(\xi_j/\xi_i)}{\vartheta(q^\beta\xi_j/\xi_i)},$$

$$J_{B_2}(\beta,\gamma;\xi)=2$$

$$\frac{\Gamma_q(1-\beta)\Gamma_q(-\gamma)\Gamma_q(1-\gamma)\Gamma_q(1-\beta-\gamma)\Gamma_q(-\beta-\gamma)\Gamma_q(-\beta-2\gamma)}{\Gamma_q(1-2\beta)\Gamma_q(1-2\gamma)\Gamma_q(-2\gamma)\Gamma_q(1-2\beta-2\gamma)}$$

$$\cdot\frac{\xi_1^{-\beta}\xi_2^{-\beta-2\gamma}\vartheta(\xi_1)\vartheta(\xi_2)\vartheta(\xi_2/\xi_1)\vartheta(\xi_2\xi_1)}{\vartheta(q^\beta\xi_1)\vartheta(q^\beta\xi_2)\vartheta(q^\gamma\xi_2/\xi_1)\vartheta(q^\gamma\xi_2\xi_1)},$$

$$J_{G_2}(\beta,\gamma;\xi)=$$

$$\frac{\Gamma_q(1-\beta)\Gamma_q(-\gamma)\Gamma_q(1-\gamma)\Gamma_q(1-\beta-\gamma)\Gamma_q(-\beta-2\gamma)\Gamma_q(-2\beta-3\gamma)}{\Gamma_q(1-2\beta)\Gamma_q(1-2\gamma)\Gamma_q(-3\gamma)\Gamma_q(1-3\beta-3\gamma)}$$

$$\cdot\frac{\xi_1^{2\gamma}\xi_2^{-2\beta-4\gamma}\vartheta(\xi_1)\vartheta(\xi_2)\vartheta(\xi_2/\xi_1)\vartheta(\xi_2\xi_1)\vartheta(\xi_2/\xi_1^2)\vartheta(\xi_2^2/\xi_1)}{\vartheta(q^\beta\xi_1)\vartheta(q^\beta\xi_2)\vartheta(q^\beta\xi_2/\xi_1)\vartheta(q^\gamma\xi_2\xi_1)\vartheta(q^\gamma\xi_2/\xi_1^2)\vartheta(q^\gamma\xi_2^2/\xi_1)}.$$

This Proposition was stated and has been proved in [8] using q-de Rham theory. After the author announced Conjecture (1) in [8], I. G. Macdonald proved it by using Poincaré series for affine root systems [9].

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