# Twist and Generalized Chebyshev Polynomials

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**1. Introduction.** In the article [2], we investigated a possible generalization of the Chebyshev polynomials $T_n(x)$, $U_n(x)$ $(n = 0,1,\cdots)$, focusing on the Diophantine equation satisfied by them: $(*)$ $T_n(x)^2 - (x^2-1)U_{n-1}(x)^2 = 1$, $n = 1,2,\cdots$. The crucial idea of [loc. cit.] was to regard this as a defining equation of a twist of a conic by itself. As a natural generalization, we considered a twist of a conic by an arbitrary hyperelliptic curve, and obtained a family of Diophantine equations which have solutions in a certain one-parameter family of polynomials in one variable. In the present article, we proceed to higher dimensional cases and consider the twist of a double cover of the affine space of dimension $N \geq 1$ by itself. As a result, we find certain families of polynomials in $N$ variables, called the *generalized Chebyshev polynomials*, which enjoy a lot of fundamental properties similar to the ones the usual Chebyshev polynomials do. The purpose of this article is to announce these properties. Details will appear elsewhere.

**2. Twist and generalized Chebyshev polynomials.** Let $k$ be an arbitrary field of characteristic $\neq 2$. Let $\boldsymbol{G}_m$ denote the multiplicative group and let $T_N$ denote the norm torus of dimension $N$. It is defined to be the kernel of the norm map $\boldsymbol{G}_m^{N+1} \to \boldsymbol{G}_m$ given by the formula: $(u_1,\cdots,u_{N+1}) \mapsto \prod_{1\leq i\leq N+1} u_i$. The norm torus $T_N$ is stable under the natural action of the symmetric group $S_{N+1}$ of degree $N+1$ on $\boldsymbol{G}_m^{N+1}$. Hence, if we denote by $A_{N+1}$ the alternating group of degree $N+1$, then we have quotient maps: $T_N \xrightarrow{p} T_N/A_{N+1} \xrightarrow{q} T_N/S_{N+1}$. We denote by $\Delta(u_1,\cdots,u_{N+1})$ the difference product $\prod_{1\leq i<j\leq N+1}(u_i-u_j)$, and by $D = D(x_1,\cdots,x_N)$ its square: $D = D(x_1,\cdots,x_N) = (\Delta(u_1,\cdots,u_{N+1}))^2$. Then the quotient $T_N/A_{N+1}$ is defined by the equation $y^2 = D(x_1,\cdots,x_N)$, where $x_k (1\leq k\leq N)$ denote the $k$-th elementary symmetric polynomial. The rational maps $p$, $q$ are given by the formulas:

$$p(u_1,\cdots,u_{N+1}) = (x_1,\cdots,x_N,\Delta),$$
$$q(x_1,\cdots,x_N,y) = (x_1,\cdots,x_N).$$

The $n$-th power endomorphism of $\boldsymbol{G}_m^{N+1}$ induces the endomorphism $[n]$ of $T_N$, and it commutes with the action of $S_{N+1}$. Therefore we have the following commutative diagram:

$$\begin{array}{ccc} T_N & \xrightarrow{[n]} & T_N \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ T_N/A_{N+1} & \xrightarrow{[n]} & T_N/A_{N+1} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ T_N/S_{N+1} & \xrightarrow{[n]} & T_N/S_{N+1}. \end{array}$$

(Here we used the same symbol $[n]$ for the induced maps). Let $T_N'$ denote the twist of $T_N/A_{N+1}$ by the quadratic extension $k(T_N/A_{N+1})/k(T_N/S_{N+1})$, where $k(X)$ denotes the rational function field of a variety $X$ defined over $k$. The twist $T_N'$ is defined over $k(T_N/S_{N+1}) \cong k(x_1,\cdots,x_N)$ and its defining equation is given by the following:

$$T_N' : D(x_1,\cdots,x_N)Y^2 = D(x_1,\cdots,X_N),$$

where the capital letters $X_1,\cdots,X_N$, $Y$ are regarded as variables (see [2] for the fundamental properties of twists). As for the set $T_N'(k(x_1,\cdots,x_N))$ of $k(x_1,\cdots,x_N)$-rational points of $T_N'$, we have the following theorem which can be proved in the same way as in [2]:

**Theorem 2.1.** *There is a natural bijection between the set $T_N'(k(x_1,\cdots,x_N))$ and the set $A = \{f \in \mathrm{Rat}_k(T_N/A_{N+1}, T_N/A_{N+1})\,;\, f\circ\iota = \iota\circ f\}$, where $\mathrm{Rat}_k(V, W)$ for $k$-varieties $V$, $W$ denotes the set of $k$-rational map of $V$ to $W$, and $\iota$ denotes the involution of $T_N/A_{N+1}$ defined by the formula $\iota(x_1,\cdots,x_N,y) = (x_1,\cdots,x_N,-y)$.*

By this theorem, the $n$-th power map $[n]$ corresponds to a $k(x_1,\cdots,x_N)$-rational point on the twist $T_N'$, which we denote by

$(t_n^{(1)}(x_1,\cdots,x_N),\cdots,t_n^{(N)}(x_1,\cdots,x_N), s_n(x_1,\cdots,x_N))$.

We call $t_n^{(k)}(x_1,\cdots,x_N)(k=1,\cdots,N)$ the *generalized Chebyshev polynomial of the first kind*, and $s_n(x,\cdots,x_N)$ *the generalized Chebyshev polynomial of the second kind*, because of the following natural generalization of $(*)$:

**Theorem 2.2.** *For any positive integer $n$, the point*

$$(X_1,\cdots,X_N,Y)=(t_n^{(1)}(x_1,\cdots,x_N),\cdots,t_n^{(N)}(x_1,\cdots,x_N),s_n(x_1,\cdots,x_N))$$

*lies on the twist*

$$T_N':D(x_1,\cdots,x_N)Y^2=D(X_1,\cdots,X_N).$$

Note that the generalized Chebyshev polynomials of the first kind are expressed as $t_n^{(k)}=m_{(n^k)}$, where $m_{(\lambda)}=\sum u_1^{a_1}\cdots u_N^{a_N}$ with $(\lambda)=(\lambda_1\cdots\lambda_N)$ a Young diagram, and the sum going through all the distinct permutations of $(\lambda_1,\cdots,\lambda_N)$. In view of this, we call $m_{(\lambda)}$ *the generalized Chebyshev polynomials.*

**3. Various properties.** The usual Chebyshev polynomials are known to have a lot of fundamental properties. Most of these can be generalized as follows.

**Theorem 3.1** (Multiplicativity). *For any positive integers $m$, $n$,*

$$\begin{cases} t_m^{(1)}(t_n^{(1)}(x_1,\cdots,x_N),\cdots,t_n^{(N)}(x_1,\cdots,x_N))=t_{mn}^{(1)}(x_1,\cdots,x_N),\\ \cdots\cdots\cdots,\\ t_m^{(N)}(t_n^{(1)}(x_1,\cdots,x_N),\cdots,t_n^{(N)}(x_1,\cdots,x_N))=t_{mn}^{(N)}(x_1,\cdots,x_N),\end{cases}$$

$$s_m(t_n^{(1)}(x_1,\cdots,x_N),\cdots,t_n^{(N)}(x_1,\cdots,x_N))\cdot s_n(x_1,\cdots,x_N)=s_{mn}(x_1,\cdots,x_N).$$

For any positive integer $m$, let $\zeta_m$ denote a primitive $m$-th root of unity, and let $\mu_m$ denote the group of $m$-th roots of unity. Then the set $Z$ of the common zeroes of $N$ polynomials $t_n^{(k)}(k=1,\cdots,N)$ is found to be

$$\Big\{(x_1,\cdots,x_N)\,;\,x_k=t_n^{(k)}(u_1,\cdots,u_{N+1}),\ \textit{with}\ u_i\in\zeta_{(N+1)n}^{i-1}\mu_n\ \textit{for}\ 1\le i\le N,\ \textit{and}\ \prod_{1\le i\le N+1}u_i=1\Big\}.$$

**Theorem 3.2** (Discrete orthogonality). *Let $N\ge 2$ and let $k$, $l$ be positive integers. If $N$ is even, then*

$$\sum_{(x_1,\cdots,x_N)\in Z}t_k^{(1)}(x_1,\cdots,x_N)t_l^{(1)}(x_1,\cdots,x_N)$$

$$=\begin{cases} n^N(N+1)^2,\ k\equiv l\equiv 0(\mathrm{mod}(N+1)n),\\ n^N(N+1),\ k+l\equiv 0(\mathrm{mod}(N+1)n)\\ \qquad\qquad \textit{and}\begin{cases}k\not\equiv 0\ (\mathrm{mod}\ n)\\ \textit{or}\\ l\not\equiv 0\ (\mathrm{mod}\ n),\end{cases}\\ 0,\qquad \textit{otherwise.}\end{cases}$$

*If $N$ is odd, then*

$$\sum_{(x_1,\cdots,x_N)\in Z}t_k^{(1)}(x_1,\cdots,x_N)t_l^{(1)}(x_1,\cdots,x_N)$$

$$=\begin{cases} n^N(N+1)^2,\ k\equiv l\equiv 0(\mathrm{mod}\,2(N+1)n),\\ n^N(N+1),\ k+l\equiv 0(\mathrm{mod}\,2(N+1)n)\\ \qquad\qquad \textit{and}\begin{cases}k\not\equiv 0\ (\mathrm{mod}\ n)\\ \textit{or}\\ l\not\equiv 0\ (\mathrm{mod}\ n),\end{cases}\\ 0,\qquad \textit{otherwise.}\end{cases}$$

From now on, we assume that the base field is $\boldsymbol{C}$, the field of complex numbers. Let $S^1$ denote the unit circle in the complex plane.

**Theorem 3.3** (Orthogonality). *Let $S$ denote the image of $\{(u_1,\cdots,u_{N+1})\in(S^1)^{N+1}\,;\ \prod_{1\le i\le N+1}u_i=1\}$ under the map $(u_1,\cdots,u_{N+1})\mapsto(x_1,\cdots,x_N)$. Then for any pair $m_{(\lambda)}$, $m_{(\mu)}$ of the generalized Chebyshev polynomials,*

$$\int_S m_{(\lambda)}(x_1,\cdots,x_N)\cdot m_{(\mu)}(x_1,\cdots,x_N)\frac{dx_1\cdots dx_N}{\sqrt{D(x_1,\cdots,x_N)}}$$

$$=\begin{cases}\dfrac{(2\pi\sqrt{-1})^N d(\lambda)}{(N+1)!},\ \textit{if}\ (\lambda_1,\lambda_2+\mu_N,\cdots,\lambda_N+\mu_2,\mu_1)=c\cdot(1,\cdots,1)\\ \qquad\qquad \textit{for some integer}\ c,\\ 0,\qquad \textit{otherwise.}\end{cases}$$

For a complex valued continuous function $f$ on $S$, let $\|f\|=\max_{x\in S}|f(x)|$.

**Theorem 3.4** (Extremal property). *For any monic polynomials $p$ of degree $n>0$ in $N$ variables $x_1,\cdots,x_N$, we have $\|p\|\ge N+1$. The equality holds if and only if $p=t_n^{(1)}$ or $p=t_n^{(N)}$.*

The following is a direct consequence of this theorem:

**Corollary** ([1], Theorem). *For polynomials $P_{n-1}(z,\bar{z})$ of degree $\le n-1$ over $\boldsymbol{C}$,*

$$\inf_{P_{n-1}}\max_{z\in S}|z^n+P_{n-1}(z,\bar{z})|=3.$$

*Moreover, the maximum is attained only when $z^n+P_{n-1}(z,\bar{z})=t_n^{(1)}(z,\bar{z})$.*

**Remark.** Our theory gives a much shorter and simpler proof of this corollary than the one in [loc. cit.]. Furthermore, if we put $N=1$ in our proof, then we obtain much simpler proof of [3], Theorem 2.1 which deals with the extremal property of the usual Chebyshev polynomials.

**Theorem 3.5** (Differential equation). *For any Young diagram $(\lambda)=(\lambda_1,\cdots,\lambda_N)$, the corresponding polynomial $m_{(\lambda)}$ is a solution of the partial differential equation*

$$\text{(3.5.1)}\quad \sum_{1\le k\le N}A_{kk}f_{x_kx_k}+\sum_{1\le k<l\le N}A_{kl}f_{x_kx_l}+\sum_{1\le k\le N}B_kf_{x_k}=(N\sum_{1\le i\le N}\lambda_i^2-2\sum_{1\le i<j\le N}\lambda_i\lambda_j)f$$

*where*

$$A_{kk}=\{-k^2+(N+1)k\}x_k^2-(N+1)$$

$$\sum_{0\le r\le k-1} (2k-2r)x_{2k-r}x_r, \quad 1 \le k \le N,$$
$$A_{kl} = \{2k((N+1)-l)\}x_k x_l - 2(N+1) \sum_{0\le r\le k-1} (k+l-2r)x_{k+l-r}x_r, \quad 1 \le k < l \le N,$$
$$B_k = k(N+1-k)x_k, \quad 1 \le k \le N.$$

Let $Q_N(\lambda) = Q_N(\lambda_1, \cdots, \lambda_N)$ denote the quadratic form $N \sum_{1\le i\le N} \lambda_i^2 - 2 \sum_{1\le i<j\le N} \lambda_i \lambda_j$.

**Theorem 3.6** (Uniqueness). *For any $E \in \boldsymbol{Z}$, the set $\{m_{(\lambda)}\,;\, Q_N(\lambda) = E\}$ of the generalized Chebyshev polynomials constitutes a base of the vector space of polynomial solutions of the partial differential equation $Df = Ef$, where $D$ denotes the differential operator appearing on the left hand side of* (3.5.1).

**Theorem 3.7** (Dimension formula). *The dimension of the space of polynomials solutions of* (3.5.1) *is*

$$\begin{cases} 0, & \text{if } E \text{ is odd,} \\ \sum_{d|E/2} \left(\frac{d}{3}\right), & \text{if } E \text{ is even and } E/2 \text{ is not a square,} \\ \sum_{d|E/2} \left(\frac{d}{3}\right) + 1, & \text{if } E \text{ is even and } E/2 \text{ is a square,} \end{cases}$$

*where $\left(-\right)$ denotes the Legendre symbol.*

**Theorem 3.8** (Numerical integration I). *For each pair $(i, j)$ of integers with $0 \le i, j \le n-1$, let $P(i, j) = (\zeta_{3n}^{3i}, \zeta_{3n}^{3j+1}, \zeta_{3n}^{-3i-3j-1})$. Then for any polynomial $f$ in two variables $x_1, x_2$ of degree $\le 2n-1$, the following numerical integration formula holds*:

$$\int_S f \frac{dx_1 dx_2}{\sqrt{D(x_1, x_2)}} = -\frac{2\pi^2}{3n^2} \sum_{0\le i,j\le n-1} f(P(i, j)).$$

**Theorem 3.9** (Numerical integration II). *For each triple $(i, j, k)$ of integers with $0 \le i, j, k \le n-1$, let $P(i, j, k) = (\zeta_{8n}^{8i+1}, \zeta_{8n}^{8j+3}, \zeta_{8n}^{8k+5}, \zeta_{8n}^{-8i-8j-8k-9})$. Then for any polynomial $f$ in three variables $x_1, x_2, x_3$ of degree $\le 2n-1$, the following numerical integration formula holds*:

$$\int_S f \frac{dx_1 dx_2 dx_3}{\sqrt{D(x_1, x_2, x_3)}} = -\frac{\pi^3}{3n^3}\sqrt{-1} \sum_{0\le i,j,k\le n-1} f(P(i, j, k)).$$

Let $t_n$ denote the self map of $S$ defined by the formula: $t_n(x) = (t_n^{(1)}(x), \cdots, t_n^{(N)}(x))$ for $x = (x_1, \cdots, x_N) \in S$. Let $\boldsymbol{B}$ denote the family of Borel subsets of $S$, and let $\mu$ be the measure defined by

$$\mu(B) = \frac{N+1}{(2\pi\sqrt{-1})^N} \int_B \frac{dx_1 \cdots dx_N}{\sqrt{D(x_1, \cdots, x_N)}}, \quad B \in \boldsymbol{B}.$$

Then we have the following:

**Theorem 3.10** (Ergodicity). *Each $t_n$ $(n \ge 1)$ preserves the measure $\mu$. Moreover, each $t_n$ $(n > 1)$ is ergodic.*

## References


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[3] T. J. Rivlin: Chebyshev Polynomials. 2nd. ed., John Wiley & Sons, New York (1990).