Transcendence of Rogers-Ramanujan Continued Fraction and Reciprocal Sums of Fibonacci Numbers

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(Communicated by Shokichi IYANAGA, M. J. A., Sept. 12, 1997)

The Rogers-Ramanujan continued fraction $RR(q)$ is defined by

$$RR(q) = 1 + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \frac{q^4}{1} + \cdots,$$

which is known to have the expansions

$$RR(q) = \frac{\sum_{k=0}^{\infty} \frac{q^{k^2}}{(1-q)(1-q^2)\dots(1-q^k)}}{\sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(1-q)(1-q^2)\dots(1-q^k)}} = \prod_{k=0}^{\infty} \frac{(1-q^{5k+2})(1-q^{5k+3})}{(1-q^{5k+1})(1-q^{5k+4})}$$

(cf. [2 ; (3.4.9)]). Irrationality measures were given by Osgood [8] and Shiokawa [9]. It is proved in [9] that, for any integer $d \geq 2$, there is a constant $C = C(d) > 0$ such that

$$\left| RR\left(\frac{1}{d}\right) - \frac{p}{q} \right| > Cq^{-2-B/\sqrt{\log q}}$$

for all integers p, q (≥ 2), where $B = \sqrt{\log d}$. Matala-Aho [5] obtained some higher degree irrationality results. An example of Theorem 1 in [5] is $RR((\sqrt{5}-1)/2) \notin \boldsymbol{Q}(\sqrt{5})$.

In this note we first prove the following.

Theorem 1. The Rogers–Ramanujan continued fraction $RR(q)$ is transcendental for any algebraic number q with $0 < |q| < 1$.

The proof is a simple application of Lemma 1 and 2 below, which are proved in the same manner as in [3]. Lemma 2 is a straightforward consequence of a recent theorem of Nesterenko on modular functions ([6] and [7]).

As usual we set for $|q| < 1$

$$E_2(q) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n,$$

$$E_4(q) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n,$$

$$E_6(q) = 1 - 540\sum_{n=1}^{\infty}\sigma_5(n)q^n,$$

where $\sigma_k(n) = \sum_{d|n} d^k$,

$$\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n),$$

and

$$\theta_3 = 1 + 2\sum_{n=1}^{\infty} q^{n^2},\ \theta = \theta_4 = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2},$$

$$\theta_2 = 2q^{1/4}\sum_{n=1}^{\infty} q^{n(n-1)}.$$

Let $\boldsymbol{K} = \boldsymbol{Q}(E_2, E_4, E_6)$.

Lemma 1 ([4]). Let $y = y(q)$ denote any one of θ_3, θ_4, and θ_2. Then the functions $\eta(q^k)$, $\eta'(q^k)$, $\eta''(q^k)$, $y(q^k)$, $y'(q^k)$, and $y''(q^k)$ are algebraic over $\boldsymbol{K}$ for every positive integer k, where "$'$" denotes the derivation $q\dfrac{d}{dq}$.

Lemma 2 ([4]). Suppose that α is an algebraic number with $0 < |\alpha| < 1$. If a nonconstant function f is algebraic over $\boldsymbol{K}$ and defined at α, then $f(\alpha)$ is transcendental.

Proof of Theorem 1. Let

$$F(q) = \frac{q^{1/5}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \frac{q^4}{1} + \cdots,$$

then

$$\frac{1}{F(q)} - F(q) - 1 = q^{1/5}\frac{\prod_{n=1}^{\infty}(1-q^{n/5})}{\prod_{n=1}^{\infty}(1-q^{5n})} = q^{2/5}\frac{\eta(q^{1/5})}{\eta(q^5)}$$

(see [1 ; p. 85]). Applying Lemma 1 and 2 to the function $f(q) = \eta(q)/\eta(q^{25})$, we see that, for any algebraic number q with $0 < |q| < 1$, $f(q)$ is transcendental, and so is $F(q)$ from the formula above.

Now we give further examples of continued fractions whose transcendence can be easily deduced from Lemma 1 and 2. For any algebraic number q with $0 < |q| < 1$, the following con-

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tinued fractions (i), (ii), and (iii) are transcendental:

(i) $$\frac{1}{1+}\frac{q}{1+q+}\frac{q^2}{1+q^2+}\frac{q^3}{1+q^3+}\cdots = \frac{\theta_2(q^{1/2})}{2q^{1/8}\theta_3(q)}$$

(see [1 ; p. 221., Entry 1 (i)]).

(ii) $$\frac{1}{1+q+}\frac{q^2}{1+q^3+}\frac{q^4}{1+q^5+}\frac{q^6}{1+q^7+}\cdots.$$

For, if we put

$$\upsilon = \frac{q^{1/2}}{1+q+}\frac{q^2}{1+q^3+}\frac{q^4}{1+q^5+}\frac{q^6}{1+q^7+}\cdots,$$

then

$$\upsilon + \frac{1}{\upsilon} = \frac{2\theta_3(q)}{\theta_2(q^2)}$$

(see [1 ; p. 221, Entry 1 (ii)]).

(iii) $$\frac{1}{1+}\frac{q+q^2}{1}\;+\;\frac{q^2+q^4}{1}\;+\;\frac{q^3+q^6}{1}\;+\cdots = \frac{\eta(q)\eta(q^6)^3}{q^{1/3}\eta(q^2)\eta(q^3)^3}$$

(see [1 ; p. 345, Entry 1]).

Next, we prove the transcendence of reciprocal sums of some binary linear recurrences. Our results below generalize those obtained in [5].

Let $k = \theta_2^2(q)/\theta_3^2(q)$. Then

$$K := \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \frac{\pi}{2}\theta_3^2(q),$$

$$E := \int_0^1 \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}dt = K + \frac{\pi^2}{k}\frac{\theta_4'(q)}{\theta_4(q)}$$

(see [2 ; (2.1.13), (2.3.17)]).

Lemma 3. Let s be a positive integer and let

$$f_{2s}(q) = \sum_{n=1}^{\infty}\frac{1}{(q^{-n}-q^n)^{2s}},$$

$$g_s(q) = \sum_{n=1}^{\infty}\frac{1}{(q^{-n}+q^n)^{s}}.$$

Then $f_{2s}(q)$, $f_{2s}(q^2)$, $g_s(q)$, and $g_s(q^2)$ are algebraic over $\boldsymbol{K}$.

Proof. By Table 1(i) in [10], $f_{2s}(q)$ can be written as a polynomial of k, K/π, E/π with rational coefficients, and so $f_{2s}(q)$ and $f_{2s}(q^2)$ are algebraic over $\boldsymbol{K}$ by Lemma 1. Similarly, $g_{2s}(q)$, $g_{2s}(q^2)$, $g_{2s-1}(q)$, and $g_{2s-1}(q^2)$ are algebraic over $\boldsymbol{K}$ by Table 1(ii), (vi) in [10].

Let α and β be algebraic with $\alpha \neq \beta$ and $|\beta| < 1$. Put

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad V_n = \alpha^n + \beta^n.$$

Theorem 2. If $\alpha\beta = \pm 1$, then the numbers

$$\sum_{n=1}^{\infty}\frac{1}{U_n^{2s}}, \qquad \sum_{n=1}^{\infty}\frac{1}{V_n^{2s}}$$

are transcendental for any positive integer s.

Theorem 3. If $\alpha\beta = 1$, then the number

$$\sum_{n=1}^{\infty}\frac{1}{V_n^{s}}$$

is transcendental for any positive integer s.

Theorem 4. If $\alpha\beta = -1$, then the number

$$\sum_{n=1}^{\infty}\frac{1}{U_{2n-1}^{2}}$$

is transcendental for any positive integer s.

Proof of Theorem 2. If $\alpha\beta = 1$, then

$$(\alpha-\beta)^{-2s}\sum_{n=1}^{\infty}\frac{1}{U_n^{2s}} = \sum_{n=1}^{\infty}\frac{1}{(\beta^{-n}-\beta^n)^{2s}} = f_{2s}(\beta),$$

$$\sum_{n=1}^{\infty}\frac{1}{V_n^{2s}} = \sum_{n=1}^{\infty}\frac{1}{(\beta^{-n}+\beta^n)^{2s}} = g_{2s}(\beta),$$

and the results follow from Lemma 2 and 3. Let $\alpha\beta = -1$. Then we have

$$\begin{aligned}(\alpha-\beta)^{-2s}\sum_{n=1}^{\infty}\frac{1}{U_n^{2s}} &= \sum_{n=1}^{\infty}\frac{1}{((-\beta)^{-n}-\beta^n)^{2s}}\\ &= \sum_{n=1}^{\infty}\frac{1}{((-\beta)^{-2n}-\beta^{2n})^{2s}}\\ &\quad+\sum_{n=1}^{\infty}\frac{1}{((-\beta)^{-(2n-1)}-\beta^{2n-1})^{2s}}\\ &= f_{2s}(\beta^2) + g_{2s}(\beta) - g_{2s}(\beta^2),\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{V_n^{2s}} &= \sum_{n=1}^{\infty}\frac{1}{((-\beta)^{-n}+\beta^n)^{2s}}\\ &= \sum_{n=1}^{\infty}\frac{1}{(\beta^{-2n}+\beta^{2n})^{2s}}\\ &\quad+\sum_{n=1}^{\infty}\frac{1}{(-\beta^{-(2n-1)}+\beta^{2n-1})^{2s}}\\ &= g_{2s}(\beta^2) + f_{2s}(\beta) - f_{2s}(\beta^2).\end{aligned}$$

Proof of Theorem 3.

$$\sum_{n=1}^{\infty}\frac{1}{V_n^{s}} = \sum_{n=1}^{\infty}\frac{1}{(\beta^{-n}+\beta^n)^{s}} = g_s(\beta).$$

Proof of Theorem 4.

$$(\alpha-\beta)^{-2s}\sum_{n=1}^{\infty}\frac{1}{U_{2n-1}^{2s}} = g_{2s}(\beta) - g_{2s}(\beta^2),$$

$$\begin{aligned}(\alpha-\beta)^{-(2s-1)}\sum_{n=1}^{\infty}\frac{1}{U_{2n-1}^{2s-1}}\\ = -\sum_{n=1}^{\infty}\frac{1}{(\beta^{-(2n-1)}+\beta^{2n-1})^{2s-1}}\end{aligned}$$

$$= - g_{2s-1}(\beta) + g_{2s-1}(\beta^2).$$

Fibonacci sequence $\{F_n\}_{n\geq 1}$ and Lucas sequence $\{L_n\}_{n\geq 1}$ are defined respectively by

$$F_{n+2} = F_{n+1} + F_n \quad (n \geq 0), \quad F_0 = 0, \quad F_1 = 1,$$
$$L_{n+2} = L_{n+1} + L_n \quad (n \geq 0), \quad L_0 = 2, \quad L_1 = 1,$$

and written as

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad L_n = \alpha^n + \beta^n \quad (n \geq 1),$$

where $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$.

Corollary. The numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_n^{2s}}, \quad \sum_{n=1}^{\infty} \frac{1}{L_n^{2s}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^{s}}, \quad \sum_{n=1}^{\infty} \frac{1}{L_{2n}^{s}}$$

are transcendental for any positive integer s.

Remark. In the special case of $s = 1$, these results are proved in [4] by direct calculation without using the tables in [10] quoted above.

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