On the Zeros of $\sum a_i expg_i$ *)

By Tuen-Wai NG and Chung-Chun YANG

Department of Mathematics, Hong Kong University of Science and Technology, Hong Kong

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Abstract: We consider entire functions of the form $f = \sum a_i e^{g_i}$, where $a_i (\not\equiv 0)$, g_i are entire functions and the orders of all a_i are less than one. If all the zeros of f are real, then $f = e^g \sum a_i e^{h_i}$, where h_i are linear functions. Using this result, we can prove that $f = a_1 e^g$ if all zeros of f are positive, which also generalizes a result obtained by A. Eremenko and L. A. Rubel.

Key words: Zero set; entire function; Borel theorem; upper half-plane; Nevanlinna theory.

1. Introduction and main results. For $i \geq 1$ and $z \in \mathbf{C}$, let $g_i(z)$ be entire functions. Let $a_i(z)$ be a non-zero entire function with $\rho(a_i) < 1$, where $\rho(g)$ denotes the order of an entire function g. Let B_1 denote the class of entire functions of the form

$$f = \sum_{i=1}^{n} a_i e^{g_i},$$

where $e^{g_i - g_j}$ is non-constant for $i \neq j$.

If all the a_i are polynomials, then such f is said to be in the class B. Clearly, B is a proper subset of B_1.

Let $Z(g)$ be the zero set of an entire function g. In [2], by using H. Cartan's theory of holomorphic curves. A. Eremenko and L. A. Rubel proved the following theorem.

Theorem A. Let $f \in B$. If $Z(f)$ is a subset of the positive real axis, except possibily finitely many points, then $f = pe^g$, where p is a polynomial and g is an entire function.

Therefore, it is natural to ask whether we can say something about the form of f if $f \in B$ and $Z(f)$ is a subset of the real axis. By adapting some of the arguments used in [6] and Nevanlinna value distribution theory for functions meromorphic in a half plane, we can answer this question even for the case $f \in B_1$. In fact, we obtained the following results.

Theorem 1. *Let $f \in B_1$. If $Z(f)$ is a subset of the real axis, except possibly finite points, then $f(z) = e^{g(z)} \sum_{i=1}^{n} a_i(z) e^{b_i z}$, where $b_i \in \mathbf{C}$, g and $a_i (\not\equiv 0)$ are entire functions with $\rho(a_i) < 1$.*

Using theorem 1, we can generalize theorem A to the following theorem.

Theorem 2. *Let $f \in B_1$. If $Z(f)$ is a subset of the positive real axis, except possibly finite points, then $f = ae^g$, where g, a are entire functions with $\rho(a) < 1$.*

Our basic tool is J. Rossi's half-plane version of Borel theorem. J. Rossi proved this version in [6] by using Tsuji's half-plane version of Nevanlinna theory. Therefore, we shall start with the basic notations of Tsuji's theory (cf. [4] and [7]); assuming the readers are familiar with the Nevanlinna Theory and its basic notations (cf. [3]).

Let $n_u(t, \infty)$ be the number of poles of f in $\{z : |z - \frac{it}{2}| \leq \frac{t}{2}, |z| \geq 1\}$, where f is meromorphic in the open upper half-plane. Define

$$N_u(r, \infty) = N_u(r, f) = \int_1^r \frac{n_u(t, \infty)}{t^2} dt,$$

$$m_u(r, \infty) = m_u(r, f)$$

$$= \frac{1}{2\pi} \int_{arcsinr^{-1}}^{\pi - arcsinr^{-1}} log^+ |f(rsin\theta e^{i\theta})| \frac{d\theta}{rsin^2\theta},$$

$$N_u(r, a) = N_u(r, \frac{1}{f-a}),\ m_u(r, a)$$

$$= m_u(r, \frac{1}{f-a})\ (a \neq \infty) \text{ and}$$

$$T_u(r, f) = m_u(r, f) + N_u(r, f).$$

Remark 1. We can also define $m_l(r, f)$, $N_l(r, f)$, $T_l(r, f)$ for functions meromorphic in the open lower half-plane in the obvious way.

Lemma 1 [4]. *Let f be meromorphic in Imz*

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$> 0(< 0)$. *Define* $m_{\alpha,\beta}(r, f) = \frac{1}{2\pi}\int_{\alpha}^{\beta} \log^+|f(re^{i\theta})|\, d\theta$. *Then*

$$\int_r^{\infty} \frac{m_{0,\pi}(t, f)}{t^3}\,dt \leq \int_r^{\infty} \frac{m_u(t, f)}{t^2}dt$$
$$(\int_r^{\infty} \frac{m_{\pi,2\pi}(t, f)}{t^3}\,dt \leq \int_r^{\infty} \frac{m_l(t, f)}{t^2}dt).$$

Lemma 2 [6]. *Let* $n \geq 2$, $S = \{f_0, \ldots, f_n\}$ *be a set of meromorphic functions such that any proper subset of* S *is linearly independent over* $\mathbf{C}$. *If* S *is linearly dependent over* $\mathbf{C}$, *then for all* r *except possibly on a set of finite measure,*

$$T_u(r) = O\{\sum_{k=0}^{n} [N_u(r, 1/f_k) + N_u(r, f_k)] + \log T_u(r) + \log r\},$$

where $T_u(r) = \max\{T_u(r, f_i/f_j) | 0 \leq i, j \leq n\}$.

Remark 2. If we replace $m_u(r, f)$, $N_u(r, f)$ and $T_u(r, f)$ by the standard Nevanlinna functionals $m(r, f)$, $N(r, f)$, $T(r, f)$ in Lemma 2, we shall obtain the original full-plane version of Borel theorem.

Lemma 3 [5]. *Let* g_i *be a transcendental entire function and* h *be a non-zero entire function such that* $T(r, h) = o(T(r, g_i))$ *as* $r \to \infty$, *for* $1 \leq i \leq n$. *Suppose* $\sum_{i=1}^{n} g_i(z) = h(z)$, *then* $\sum_{i=1}^{n}\delta(0, g_i) \leq n - 1$.

Lemma 4. *For* $n \geq 2$ *and each* $1 \leq i \leq n$, *let* a_i *denote a non-zero entire function with* $\rho(a_i) < 1$ *and* b_i *be a non-zero complex number. Then, there exists a positive constant* A *such that for sufficiently large* r, $T(r, a_1(z) + \sum_{i=2}^{n} a_i(z)e^{b_i z}) \geq Ar$.

The proof of Lemma 4. It is not difficult to prove for $n = 2$. Assume $n \geq 3$. Let $g(z) = a_1(z) + \sum_{i=2}^{n} a_i(z)e^{b_i z}$ and $G(z) = a_1(z) + \sum_{i=2}^{n-1} a_i(z) e^{b_i z}$. Then $T(r, G) = O(r)$ for large r. From $g = G + a_n e^{b_n z}$ and a simple calculation give

$$(a_n b_n + a'_n - a_n G'/G)e^{b_n z} = g' - gG'/G.$$

It is well-known that (for large r) $T(r, G'/G) = o(T(r, G))$ and $T(r, g') \leq AT(Br, g)$, where $A, B \geq 1$. Hence,

$$\frac{1}{\pi}|b_n| r \sim T(r, e^{b_n z}) \leq T(r, g' - gG'/G) + T(r, a_n b_n + a'_n - a_n G'/G) + O(1) \leq CT(Br, g) + o(r).$$

Therefore, for large r, $T(r, g) \geq Ar$ for some suitable positive constant A.

2. Proofs of Theorems. The proof of Theorem 1. $f \in B_1$ implies that $f = \sum_{i=1}^{n} a_i \exp g_i$, where $a_i (\not\equiv 0)$, g_i are entire functions with $T(r, a_i) = O(r^{\epsilon})$ for some fixed positive $\epsilon < 1$.

If $n = 1$, then we are done. For $n \geq 2$, suppose that $\exp(g_i - g_j)$ is non-constant for $i \neq j$. From these and using the full-plane version of Borel theorem, we can show that the functions $f_i = a_i \exp g_i$ are linearly independent. Set $f_0 = f$, then the set $\{f_0, \ldots f_n\}$ will satisfies the independence criteria of Lemma 2.

Suppose that $Z(f)$ is a subset of the real axis, except possibly finite points. Then, $N_u(r, 1/f_0) = O(\log r)$. For $1 \leq i \leq n$, we also have $N_u(r, 1/f_i) = O(r^{\epsilon})$, since

$$N_u(r, 1/f_i) = \int_1^r \frac{n_u(t, 1/a_i)}{t^2}dt \leq \int_1^r \frac{n(t, 1/a_i)}{t}dt = N(r, 1/a_i) + O(1) = O(r^{\epsilon}).$$

It follows from Lemma 2 that $T_u(r) = O(r^{\epsilon})$ and hence $T_u(r, f_i/f_j) = O(r^{\epsilon})$ for all i, j. Since $T_u(r, f_i/f_j) = N_u(r, f_i/f_j) + m_u(r, f_i/f_j)$, we also have $m_u(r, f_i/f_j) = O(r^{\epsilon})$. Similarly, $m_l(r, f_i/f_j) = O(r^{\epsilon})$. Now,

$$T(t, f_j/f_i) = N(t, f_i/f_j) + m(t, f_i/f_j) = O(t^{\epsilon}) + m_{0,\pi}(t, f_i/f_j) + m_{\pi,2\pi}(t, f_i/f_j).$$

Then by Lemma 1, we have

$$T(r, f_i/f_j)O(1/r^2) \leq \int_r^{\infty} \frac{T(t, f_i/f_j)}{t^3}dt = O(r^{-\epsilon}).$$

Consequently, $T(r, f_i/f_j) = O(r^{2-\epsilon})$. This implies that the order of $\exp(g_i - g_j)$ is less than 2 and hence equal to one.

Now, $f = e^{g_1}(a_1 + \sum_{i=2}^{n} a_i e^{g_i - g_1})$, where $g_i - g_1$ is linear for $2 \leq i \leq n$. This also completes the proof.

The proof of Theorem 2. Let $f \in B_1$ such that $Z(f)$ is a subset of the positive real axis, possibly finite points. By Theorem 1, either (i) $f = ae^{g}$ or (ii) $f(z) = e^{g(z)}(a_1(z) + \sum_{i=2}^{n} a_i(z)e^{b_i z}$, where g, $a_i (\not\equiv 0)$ are entire functions, $\rho(a_i) < 1$ and the b_i's are non-zero complex numbers. We only need to consider case (ii).

Let $G(z) = a_1(z) + \sum_{i=2}^{n} a_i(z)e^{b_i z}$, $h = -a_1$, $g_1 = -G$, $g_i(z) = a_i(z)e^{b_i z}$ for $2 \leq i \leq n$. Then $Z(G) = Z(f)$, $\sum_{i=1}^{n} g_i(z) = h(z)$, and $T(r, h) = o(T(r, g_i))$ as r tends to infinity for $1 \leq i \leq n$. By Lemma 3, $\sum_{i=1}^{n}\delta(0, g_i) \leq n - 1$. Since $\delta(0, g_i) = 1$ for $i \geq 2$, it follows that $\delta(0, G) = \delta(0, g_1) = 0$.

Hence there exists an unbounded sequence $\{r_i\}$ such that $N(r_i, 0, G) \geq \frac{1}{2}T(r_i, G)$. By Lemma 4,

$$\int_{r_i}^{\infty} \frac{N(t,0,G)}{t^2}dt \geq \int_{r_i}^{\infty} \frac{N(r_i,0,G)}{t^2}dt$$

$$\geq \int_{r_i}^{\infty} \frac{1}{2} \frac{T(r_i,G)}{t^2} dt \geq \int_{r_i}^{\infty} \frac{1}{2} \frac{Ar_i}{t^2} dt = \frac{1}{2} A > 0.$$

Therefore, $\int_0^{\infty} \frac{N(t,0,G)}{t^2} dt$ does not converage and hence the genus of G is at least one. Now, G is an entire function of finite order with a genus at least one, which has at most finitely many non-positive zeros. By a result of A. Edrei and W. Fuchs [1], $\delta(0, G) > 0$, which is a contradiction. Hence f must equal to the required form, ae^g.

Remark 3. It is obvious that Theorem A can also be derived from the present arguments by assuming that the coefficients a_i (z) are polynomials in Theorem 2.

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