Construction of High-rank Elliptic Curves with a Non-trivial Rational Point of Order 2

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In [2], Nagao constructed a family of infinitely many elliptic curves over $\boldsymbol{Q}$ with a non-trivial rational 2-torsion point and with rank ≥ 6. Also Fermigier gave in [3] an example of an elliptic curve over $\boldsymbol{Q}(t)$ with non-constant modular invariant, of rank at least 8, with a non-trivial 2-torsion point, and showed as a corollary that there are infinitely many elliptic curves of rank at least 8 with a rational 2-torsion point.

In this paper we improve these results, and prove the,

Theorem. *There are infinitely many elliptic curves over $\boldsymbol{Q}$, of rank at least* 9 *with a non-trivial rational* 2*-torsion point.*

1. We consider the following elliptic curve ε over $\boldsymbol{Q}(t)$. The construction is similar as in [2].

$\varepsilon : y^2 = Ax^4 + Bx^2 + C$

where $A = t^2 + 686$,

$B = -2t^4 + 216t^2 - 68257$, and

$C = t^6 - 326t^4 + 30529t^2 + 1568196$.

There are following points on ε,

$P_1 = (t + 5,\ 26t^2 + 315t - 539)$,

$P_2 = (-t + 5,\ 26t^2 - 315t - 539)$,

$P_3 = (t + 9,\ 15(2t^2 + 35t + 49))$,

$P_4 = (-t + 9,\ 15(2t^2 - 35t + 49))$,

$P_5 = (t + 16,\ 10(4t^2 + 84t + 539))$, and

$P_6 = (-t + 16,\ 10(4t^2 - 84t + 539))$.

There is still another point on ε where

$$P_7 = \left(\frac{-5t + 7}{7},\ \frac{-24t^3 - 70t^2 + 1911t + 60025}{49}\right).$$

Now we specialize $t = \dfrac{-s^2 + 32}{2s}$, then we have one more point on ε where

$$P_8 = \left(\frac{s^2 + 32}{2s},\ \frac{6(s^4 - 15s^2 + 1024)}{s^2}\right).$$

To get more points, we consider ε over the function field of the following elliptic curve over $\boldsymbol{Q}$ with a positive rank.

$$C : q^2 = p(p + 11520)(p + 11648).$$

The rank of C is positive as (4480, 1075200) is on C and this point is of infinite order in the Mordell-Weil group of C, by the Lutz-Nagell theorem.

Let $\boldsymbol{Q}(C)$ be the function field of C, we now consider ε over $\boldsymbol{Q}(C)$ by specializing $s = \dfrac{q}{2p}$.

Then we have the point $P_9 = (x_9, y_9)$ on ε where

$x_9 = (134184960 - p^2)/(4q)$ and

$y_9 = 3(18005603490201600 + 13047608770560p + 1984393728p^2 + 97236p^3 + p^4)/(2q^2)$.

2. Now we consider the following elliptic curve. $\varepsilon' : y^2 = x(Ax^2 + Bx + C)$.

Let $P_i = (x_i, y_i)$, where $1 \leq i \leq 9$, be the above points on ε, then $Q_i = (x_i^2, x_i y_i)$ are on ε'.

Proposition. $Q_1, \ldots, Q_9$ are independent points.

Proof. We specialize $(p, q) = (4480, 1075200)$.

Then we have 9 rational points $R_1, \ldots, R_9$ obtained from $Q_1, \ldots, Q_9$. By using calculation system PARI, we see that the determinant of the matrix $(\langle R_i, R_j \rangle)$ $(1 \leq i, j \leq 9)$ associated to the canonical height is 14736043141.66. Since this determinant is non-zero, we see $Q_1, \ldots, Q_9$ are independent. Q. E. D.

Now this Proposition and Theorem 20.3 in [1] establishes our Theorem.

References

[1] J. H. Silverman: The arithmetic of elliptic curves. Graduate Texts in Math., vol. 106, Springer-Verlag, New York (1986).

[2] K. Nagao: Construction of high-rank elliptic curves with a non trivial torsion point. Math. Comp., **66**, 411–415 (1997).

[3] S. Fermigier: Exemples de courbes elliptiques de grand rang sur $\boldsymbol{Q}(t)$ et sur $\boldsymbol{Q}$ possédant des points d'order 2. C. R. Acad. Sci. Paris, **322**, ser. 1, 949–952 (1996).