Equidimensional Relatively Stable Reductive Algebraic Groups with Simple Commutator Subgroups

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§1. Introduction. Algebraic varieties including algebraic groups are assumed to be defined over the complex number field $\boldsymbol{C}$. Let G always stand for a reductive algebraic group. An algebraic action of G on an affine variety X (abbr. (X, G)) is said to be cofree (resp. equidimensional), if the coordinate ring $\mathcal{O}(X)$ of X is $\mathcal{O}(X)^G$-free (resp. the quotient map $X \to X//G$ to the algebraic quotient $X//G = \operatorname{Specm}(\mathcal{O}(X)^G)$ is equidimensional). For any G, we denote by G' the commutator subgroup of G and use any of the notations ρ, (ρ, G) or (V, G) to denote a finite-dimensional linear representation $\rho: G \to GL(V)$ over $\boldsymbol{C}$.

Definition 1. (V, G) is defined to be *relatively stable*, if the natural action of G on $V//G'$ is stable (i.e., $V//G'$ contains a non-empty open set consisting of closed G-orbits).

Difinition 2. (V, G) is said to be *relatively quasi-stable*, if there is a relatively stable G-submodule U of V such that the inclusion $U \hookrightarrow V$ induces $U//G \cong V//G$.

As a partial affirmative answer to the conjecture brought up by V. L. Popov [6] and [7] and by V. G. Kac [2] on equidimensional representations, we obtain the following:

Theorem 1. *Suppose that G is a connected reductive algebraic group whose commutator subgroup is a simple algebraic group. If a finite-dimensional linear representation (V, G) is equidimensional and relatively quasi-stable, then it is cofree.*

In the case that G' is non-orthogonal symplectic, it has already been obtained as a somewhat general result of [4]. When G itself is simple, the result similar to this is shown by Popov [6] and O. M. Adamovich [1], and, moreover, cofree representations of G are determined by [6] and G. W. Schwarz [9].

The purpose of this paper is to explain the summary of our results concerning Theorem 1.

§2. Preliminaries. The L-module (V, G) extended by a morphism $\mu: L \to G$ is donoted to V_μ. In the case that $G = G_1 \times G_2$, any representation δ of the component G_i may be identified with $\delta_{G \to G_i}$. For a representation φ of G and a non-negative integer m, $m\varphi$ denotes the direct sum of m copies of φ and φ^* denotes the dual of φ. Especially if G is connected, for irreducible φ and ψ, φ^i and $\varphi^i\psi^j$ $(i, j \in \boldsymbol{N})$ denote, respectively, the irreducible component of heighest weight in $\operatorname{Sym}^i(\varphi)$ and in $\operatorname{Sym}^i(\varphi) \otimes \operatorname{Sym}^j(\psi)$. For any simple, connected and simply-connected algebraic group of rank r, let $\Phi_1, \ldots, \Phi_r$ denote its basic representations whose orderings are the same as those in [10].

Definition 3. (V, G) is defined to be *relatively equidimensional*, if the action $(V//G', G)$ is equidimensional.

Definition 4. (V, G) is defined to be *relatively irredundant along trivial parts*, if $(V//G', G)$ is non-trivial and ($V^{G'} = \{0\}$ or) $G|_{V//G'}$ is never equal to the inner direct product

$$(\cap_{z \in (V/U)//G'} (G|_{V//G'})_z) \times (\cap_{z \in U} (G|_{V//G'})_z),$$

for any non-zero subspace U of $V^{G'}$.

Let $\mathfrak{X}(G)$ denote the rational linear character group of G. A sequence $(\chi_1, \ldots, \chi_m)$ in the rational character group $\mathfrak{X}(G)$ is said to be *UPR*, if $\operatorname{rk} < \chi_1, \ldots, \chi_m > = m - 1$ and $\Pi_{i=1}^m \chi_i^{a_i} = 1$ for some $0 < a_i \in \boldsymbol{Q}$.

Example 1. Suppose that G is connected and is a direct product of a simple connected and

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simply-connected group G' and a torus T and that the ineffective part of T on V is finite.

(1) Suppose that $(V, G)|_{G'}$ contains an irreducible representation of G' with finite principal subgroups. Then (V, G) is relatively stable, relatively equidimensional and relatively irredundant along trivial parts if and only if it can be identified with

$$(\Phi_1^3\chi_0 \oplus \chi_1 \oplus \ldots \oplus \chi_m, \mathbf{A}_1 \times \boldsymbol{C}^{*m})$$

for an $m \in \boldsymbol{N}$ and a *UPR* sequence $(\chi_0^4, \chi_1, \ldots, \chi_m)$ in $\mathfrak{X}(\boldsymbol{C}^{*m})$. The *only if* part of this assertion is derived from Lemma 1.5 of [4].

(2) Suppose that $(V, G)|_{G'}$ contains the adjoint representation (Ad, G'). Then (V, G) is relatively stable, relatively equidimensional and relatively irredundant along trivial parts if and only if, up to an automorphism, it can be identified with one of the following representations;

$$(\mathrm{Ad} \oplus \Phi_1\chi \oplus \Phi_1^*\chi^{-1}, \mathbf{A}_r \times \boldsymbol{C}^*)$$

for a nontrivial $\chi \in \mathfrak{X}(\boldsymbol{C}^*)$;

$$(\mathrm{Ad} \oplus \Phi_1\chi_0 \oplus \chi_1 \oplus \cdots \oplus \chi_m, \mathbf{A}_r \times \boldsymbol{C}^{*m})$$

for an $m \in \boldsymbol{N}$ and a *UPR* sequence $(\chi_0^{r+1}, \chi_1, \ldots, \chi_m)$ in $\mathfrak{X}(\boldsymbol{C}^{*m})$;

$$(\mathrm{Ad}\chi \oplus \mathrm{Ad}\chi^{-1}, \mathbf{A}_1 \times \boldsymbol{C}^*)$$

for a nontrivial $\chi \in \mathfrak{X}(\boldsymbol{C}^*)$;

$$(\mathrm{Ad}\chi_0 \oplus \Phi_1\chi_1 \oplus \Phi_1\chi_2, \mathbf{A}_1 \times \boldsymbol{C}^{*2})$$

for $\chi_i \in \mathfrak{X}(\boldsymbol{C}^{*2})$ such that $\chi_0\chi_1\chi_2 = 1$ and $\mathrm{rk} < \chi_0, \chi_1, \chi_2 > = 2$; or with

$$(\mathrm{Ad}\chi_0 \oplus \Phi_1\chi_1 \oplus \chi_2 \oplus \ldots \oplus \chi_m, \mathbf{A}_1 \times \boldsymbol{C}^{*m})$$

for a *UPI* sequence $(\chi_0^2, \chi_0\chi_1^2, \chi_2, \ldots, \chi_m)$ in $\mathfrak{X}(\boldsymbol{C}^{*m})$. The *only if* part of this assertion in the case where $\mathrm{rk}_{ss}G \geq 2$ is derived from Lemma 1.4 and Lemma 1.5 of [4].

§3. Classifications. In Tables I and II, let H be a simple, connected and simply-connected group, W a non-zero representation of H and m, n nonnegative integers.

Theorem 2. (1) *Suppose that G' is a simple algebraic group of one of the following types;* $\mathbf{B}_r$ $(r \geq 2)$, $\mathbf{D}_r$ $(r \geq 4)$, $\mathbf{G}_2$, $\mathbf{F}_4$, and $\mathbf{E}_r$ $(r = 6, 7, 8)$. *If (V, G) is a relatively equidimensional, relatively stable and relatively irredundant along trivial parts representation, then, for a covering morphism ν: $H \to G'$, the representation $((V/V^{G'})_\nu, H)$ is isomorphic to one of (W, H)'s listed in Table I.*

(2) *For any nontrivial (W, H) listed in Table I, there is a relatively equidimensional, relatively stable and relatively irredundant along trivial parts representation (V, G) such that $G' = H$ and $(V/V^{G'}, G')$ is isomorphic to (W, H).*

Table I

H	W
B_2	$m\Phi_1 \oplus n\Phi_2$; $m = 1, 2$, $n \leq 2$
B_2	$m\Phi_2$; $2 \leq m \leq 4$
B_3	$m\Phi_1 \oplus n\Phi_3$; $m, n \leq 2$
B_4	$m\Phi_1 \oplus \Phi_4$; $m \leq 1$
B_4	$2\Phi_4$
B_5	$m\Phi_1 \oplus \Phi_5$; $m \leq 1$
B_r $(r \geq 4)$	$m\Phi_1$; $m \leq 2$
D_4	$m\Phi_1 \oplus n\Phi_3$; $m \leq 3$, $n = 1, 2$
D_4	$m\Phi_1 \oplus \Phi_3 \oplus \Phi_4$; $m \leq 3$
D_5	$\Phi_1 \oplus m\Phi_4$; $m = 1, 2$
D_5	$2\Phi_4$
D_5	$\Phi_1 \oplus \Phi_4 \oplus \Phi_5$
D_6	$m\Phi_1 \oplus \Phi_5$; $m \leq 2$
D_6	$\Phi_5 \oplus \Phi_6$
D_7	$m\Phi_1 \oplus \Phi_6$; $m \leq 1$
D_r $(r \geq 4)$	$m\Phi_1$; $m \leq 2$
G_2	$m\Phi_1$; $m \leq 2$
F_4	Φ_1
E_6	$m\Phi_1 \oplus \Phi_5$; $m \leq 1$
E_7	Φ_1

For G in Theorem 2, we can explicitly determine relatively stable and relatively equidimensional representations.

Remark 1. In the case where a simple G' is of type $\mathbf{C}_r$ $(r \geq 3)$, a classification similar to Theorem 2 can be derived from [4].

Theorem 3. *Suppose that G' is a simple algebraic group of type $\mathbf{A}_r$. If (V, G) is an equidimensional, relatively stable and relatively irredundant along trivial parts representation, then, for a covering morphism $\nu: H \to G'$, the representation $((V/V^{G'})_\nu, H)$ is isomorphic to a subrepresentation of one of (W, H)'s listed in Table II.*

Since the algebraic torus G/G' acts on $V/\!/G'$, the result in [5] plays an important role in the proofs of Theorem 2 and Theorem 3.

Remark 2. Some (W, H)'s listed in those tables are not cofree and, what is worse, are not coregular (cf. [8] and [9]).

Proposition 1 (cf. [9] and [12]). *Suppose that U is a G-submodule of (V, G) such that the inclusion $U \hookrightarrow V$ induces $U/\!/G \cong V/\!/G$. Then (V, G) is equidimensional (resp. cofree) if and only if so is (U, G). If (V, G) is (relatively) equidimensional, then so is (W, G)* for any G-submodule W.

For an algebraic action (X, G) on an affine

Table II

$H = \mathrm{A}_r$	W
$r = 1$	$\Phi_1 \oplus \Phi_1^3$
$r = 1$	$4\Phi_1$
$r \geq 1$	$2\Phi_1 \oplus \Phi_1^2$
$r \geq 1$	$2\Phi_1^2$
$r \geq 2$	$\Phi_1^2 \oplus \Phi_1^{*2}$
$r \geq 3$	$\Phi_1 \oplus \Phi_1^2 \oplus \Phi_2$
$r \geq 3$	$\Phi_1^2 \oplus \Phi_2^* \oplus \Phi_1^*$
$r \geq 2$	$\Phi_1 \oplus \Phi_1^2 \oplus \max\{[r/2], 2\}\Phi_1^*$
$r \geq 3$	$\Phi_1^2 \oplus ([(r+1)/2]+1)\Phi_1^*$
r odd ≥ 5	$\Phi_1 \oplus \Phi_1^2 \oplus \Phi_2^*$
r odd ≥ 3	$\Phi_1 \oplus 2\Phi_2 \oplus 2\Phi_1^*$
r odd ≥ 3	$2\Phi_1 \oplus 2\Phi_2$
r even ≥ 4	$2\Phi_1 \oplus 2\Phi_2 \oplus \Phi_1^*$
r even ≥ 4	$2\Phi_1 \oplus \Phi_2 \oplus \Phi_2^* \oplus \Phi_1^*$
r odd ≥ 5	$2\Phi_1 \oplus \Phi_2 \oplus \Phi_2^*$
r odd ≥ 5	$\Phi_1 \oplus \Phi_2 \oplus \Phi_2^* \oplus \Phi_1^*$
$r = 4$	$2\Phi_2 \oplus \Phi_2^*$
$r = 5$	$\Phi_1 \oplus 3\Phi_2$
$r \geq 3$	$4\Phi_1 \oplus \Phi_2$
$r \geq 7$	$3\Phi_1 \oplus \Phi_2 \oplus (r-3)\Phi_1^*$
$r = 4$	$3\Phi_1 \oplus \Phi_2 \oplus 3\Phi_1^*$
$r = 5, 6$	$3\Phi_1 \oplus \Phi_2 \oplus 4\Phi_1^*$
$r = 3$	$2\Phi_1 \oplus \Phi_2 \oplus 3\Phi_1^*$
$r = 5$	$\Phi_1^2 \oplus \Phi_3$
$r = 5$	$\Phi_2 \oplus \Phi_3$
$r = 5$	$3\Phi_1 \oplus \Phi_3$
$r = 5$	$2\Phi_1 \oplus \Phi_3 \oplus 2\Phi_1^*$
$r = 6$	$2\Phi_1 \oplus \Phi_3 \oplus \Phi_1^*$
$r = 6$	$\Phi_1 \oplus \Phi_3 \oplus 2\Phi_1^*$
$r = 6$	$\Phi_3 \oplus 3\Phi_1^*$
$r = 7$	$\Phi_1 \oplus \Phi_3$
$r = 7$	$\Phi_3 \oplus \Phi_1^*$
$r \geq 4$	$\Phi_2 \oplus r\Phi_1^*$
$r \geq 2$	$(r+2)\Phi_1 \oplus \Phi_1^*$
$r \geq 2$	$r\Phi_1 \oplus r\Phi_1^*$
$r \geq 2$	$\Phi_1 \oplus \Phi_1^* \oplus \Phi_1\Phi_1^*$

variety X, we denote by $\mathcal{O}(X)^{St(X,G)}$ the subalgebra of $\mathcal{O}(X)$ generated by the union of modules $\mathcal{O}(X)_\chi$'s of invariants in $\mathcal{O}(X)$ relative to χ for all $\chi \in \{\mu \in \mathfrak{X}(G) \mid \mathcal{O}(X)_\mu \cap \mathcal{O}(X)^G \neq \{0\}\}$, which is denoted to $\mathfrak{X}_X(G)$. Then, by [5], we have $\mathcal{O}(X)^{St(X,G)} = \mathcal{O}(X//(\cap_{\chi \in \mathfrak{X}_X(G)} \mathrm{Ker}\chi))$.

Proposition 2. (Compare [3]). *Suppose that the canonical morphism $V//(\cap_{\chi \in \mathfrak{X}_V(G)} \mathrm{Ker}\chi) \to V//G$ is equidimensional. If* $\dim((V/V^{G'})//G) \leq 2$, *then (V, G) is coregular.*

Our main result, Theorem 1, is an (indirect) consequence of the results in this section, [8], [9] and further computations.

We have the following criterion on quasi-stability of algebraic actions:

Proposition 3. *For (V, G), it is relatively quasi-stable if and only if there is a G-submodule U of V such that the inclusion $U \hookrightarrow V$ induces $\mathcal{O}(U)^{G'} \cong \mathcal{O}(V)^{St(V,G)}$.*

Example 2. In general, by Proposition 3, we see that (V, G) is relatively quasi-stable, if
$$\dim((V^{Z(G)^0} \oplus \psi)//G') > \dim((V^{Z(G)^0})//G'),$$
for any irreducible component (ψ, G) of $(V/V^{Z(G)^0}, G)$. For example, we suppose that the commutator subgroup G' is a simple algebraic group of one of the following types; $\mathbf{B}_r$ $(r \geq 3)$, $\mathbf{D}_4$, $\mathbf{D}_r$ $(r \geq 6)$, $\mathbf{G}_2$, $\mathbf{F}_4$, and $\mathbf{E}_r$ $(r = 6, 7, 8)$. Then, since the above inequality always holds, all representations of G are relatively quasi-stable. Consequently, in this case, all equidimensional representations of G are cofree.

Remark 3. Suppose that $\mathrm{rk}_{ss} G = 1$. Equidimensional representations of G are cofree, although it is not required that they are relatively quasi-stable.

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