About splitting numbers

By Toshio SUZUKI

Department of Mathematics and Information Sciences, Osaka Prefecture University
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The main purpose of this paper is to present a proof of the following theorem with some expositions for working mathematician's who are not set theorists.

Theorem 1. *Suppose that κ is an uncountable regular cardinal. Then, the splitting number of κ is strictly larger than κ if and only if κ is a weakly compact cardinal.*

In August 1992, the author reported a sketch of a proof of Theorem 1 and that of Proposition 2, in a meeting at the Research Institute of Mathematical Sciences, Kyoto University [10]. Since then, a previous version of our current paper has been circulated among some set theorists. In October 1992, Zapletal [11], who was a graduate student of the Pennsylvania State University at that time, informed us that he improved our results. Nowadays, the theory of splitting number at uncountable cardinals relates to various branches of set theory such as inner model theory and Shelah's pcf theory. Our current paper is the final version of the preprint with the same title, which appears at the list of references of Zapletal's paper [11].

It is not hard to see that our argument gives an alternative proof of the following fact due to Johnson [4, Corollary 2]: "Suppose that κ is an uncountable regular cardinal. Then, κ is weakly compact iff I_κ is WC iff I_κ is (κ, κ)-distributive". Johnson showed this fact by using forcing. In [11, Lemma 4], Zapletal cited our Theorem 1, and he presented a modified proof that uses ultrapowers. On the other hand, our proof of Theorem 1 is purely combinatorial.

§1. The classical splitting number. Not only the cardinality of the continuum, but also some invariants of the continuum have applications in studies of topological issues; the splitting number s is one of such invariants [2]. To see the definition of the splitting number, let us define some auxiliary concepts. For a set X and a cardinal number κ, the collection of all sets that satisfy the following two conditions is denoted by $[X]^\kappa$: (1) it is a subset of X; (2) it has cardinality κ. In accordance with usual manner of set theory, we identify $\aleph_0$ with $\boldsymbol{N}$, the collection of all natural numbers (including zero). In the following, ω stands for $\aleph_0 (= \boldsymbol{N})$. Moreover, ω_α $(= \aleph_\alpha)$ denotes the α-th cardinal number; $\omega_0 = \omega$ and ω_1 is the least uncountable cardinal. For a cardinal number κ, 2^κ denotes the cardinality of the power set of κ; thus, 2^ω is the cardinality of the continuum. If $\kappa = \omega_\alpha$, then κ^+ stands for $\omega_{\alpha+1}$. ZFC denotes the usual formal system of set theory i.e. Zermelo-Fraenkel set theory with axiom of choice. Jech's book [3] is a standard textbook of basic concepts of set theory.

$\mathcal{S} \subseteq [\omega]^\omega$ is called *a splitting family* if for each $X \in [\omega]^\omega$, there exists an $A \in \mathcal{S}$ such that $|X \cap A| = |X \setminus A| = \omega$, where $|X|$ denotes the cardinality of X. The *splitting number* s is the minimum cardinality of a splitting family: $s =_{def} \min\{|\mathcal{S}| : \mathcal{S} \subseteq [\omega]^\omega$ is a splitting family$\}$.

The second assertion of the following fact is shown by forcing.

Fact 1 ([2]).

1. $\omega_1 \leq s \leq 2^\omega$.
2. *The consistency of ZFC implies the consistency of the following: "$ZFC + \omega_2 \leq s$".*

§2. The splitting number on ω_1. In 1991, Shizuo Kamo at Osaka Prefecture University defined the splitting number on an arbitrary infinite cardinal number κ as follows. $\mathcal{S} \subseteq [\kappa]^\kappa$ is called a *splitting family on κ* if for all $X \in [\kappa]^\kappa$ there exists an $A \in \mathcal{S}$ such that $|X \cap A| = |X \setminus A| = \kappa$. *The splitting number of κ*, which is denoted by $s(\kappa)$, is the minimum cardinality of a splitting family on κ: $s(\kappa) =_{def} \min\{|\mathcal{S}| : \mathcal{S} \subseteq [\kappa]^\kappa$ is a splitting family$\}$. Clearly, the classical splitting number s is $s(\omega)$. His graduate student Motoyoshi tried to show, under the assumption that κ is an uncountable regular cardinal, the fol-

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lowing two assertions which are analogues of Fact 1. (I) "$\kappa^+ \leq s(\kappa)$". (II) "There exists a notion of forcing which forces the following assertion: 'κ is an uncountable regular cardinal and $\kappa^{++} \leq s(\kappa)$'". However, he found a counter example of (I), especially $s(\omega_1) = \omega_0$. In the sequel, he showed that even a weak form of (I) is equiconsistent to a strong axiom of infinity.

Fact 2 ([9]). *Suppose that κ is an uncountable regular cardinal. Then, $\kappa \leq s(\kappa)$ if and only if κ is a strongly inaccessible cardinal.*

Therefore, we can not prove the assertion (II) in ZFC provided that ZFC is consistent.

Remark. The assertion that there exists a strongly inaccessible cardinal is a strong axiom of infinity. More precisely, the assertion implies usual axiom of infinity, and the assertion is strictly stronger than usual axiom of infinity with respect to consistency strength. Then, why we are interested in adding strong axioms of infinity to our formal system of set theory, or why we are not necessarily satisfied with usual ZFC set theory? Let us explain it by an example. ZFC is too weak to settle some problems about subsets of Euclidean space that are not Borel subsets. That is, assume that n is a sufficiently large natural number and G_1 is a Borel subset of the n-dimensional Euclidean space $\boldsymbol{R}^n$. Let us take a positive integer m that is slightly smaller than n and suppose that G_2 is a projection of G_1 to $\boldsymbol{R}^m$. Next, let G_3 be the complement of G_2 in $\boldsymbol{R}^m$. Again, we take a positive integer l that is slightly smaller than m and suppose that G_4 is a projection of G_3 to $\boldsymbol{R}^l$. Suppose that we iterate such a procedure finite times, and that G_k is a resulting subset of $\boldsymbol{R}^j$. Then, in general, we can not prove in ZFC that G_k is Lebesgue measurable in $\boldsymbol{R}^j$. However, if we add certain strong axiom of infinity to ZFC, then we can remove such a pathology [8]. Strong axioms of infinity are often called large cardinal axioms, because they are of the form "such and such large cardinal exists in our universe of set theory". We can linearly order most of known large cardinal axioms in accordance with their consistency strength. Strongly inaccessible cardinals, weakly compact cardinals and measurable cardinals are typical examples of the notion of large cardinals. For definitions of these cardinals, see [6] or [3]. As is illustrated by the above example about sets that are slightly more complicated than Borel sets, behavior of the real line under a strong axiom of infinity is one major subject of modern set theory.

Although the classical splitting number was a subject of set-theoretic topology, Fact 2 suggests a relationship between splitting numbers and strong axioms of infinity. In fact, it turned out that there exists a close connection between splitting numbers and strong axioms of infinity as we shall see in the next section.

§3. Strong axioms of infinity. In this section, we consider the consistency strength of the following two assertions. (I′) "For some uncountable regular cardinal κ, we have $\kappa^+ \leq s(\kappa)$". (II′) "For some uncountable regular cardinal κ, we have $\kappa^{++} \leq s(\kappa)$". The following is a proof of Theorem 1.

Proof of Theorem 1. Suppose that κ is an uncountable regular cardinal.

Lemma 1. *If we have $\kappa^+ \leq s(\kappa)$, then κ is weakly compact.*

Proof of Lemma 1. Suppose $\kappa^+ \leq s(\kappa)$. By Fact 2, κ is strongly inaccessible. Assume for a contradiction that κ is not weakly compact. Then, there exists a κ-Aronszajn tree $(T, \prec)$, by [3, Lemma 29. 6]. We may assume that $T = \kappa$ and $(T, \prec)$ is well-pruned; the reader may find the definitions of a κ-Aronszajn tree and a well-pruned tree in [7, p. 69; p. 71]. For each $t \in T$, let $S_t = \{u \in T : t \prec u\}$. Let $\mathcal{S}$ be defined by: $\mathcal{S} =_{def} \{S_t : t \in T \text{ and } |S_t| = \kappa\}$. Let us fix an arbitrary $X \in [T]^\kappa$.

Claim. *There exists an $\alpha < \kappa$ with the following property: the α-th level of T has two distinct members u and v such that $|X \cap S_u| = |X \cap S_v| = \kappa$.*

Proof of Claim. Suppose that Claim fails. Since κ is strongly inaccessible, we can construct a branch of size κ in T by induction. This contradicts the fact that T is a κ-Aronszajn tree. (Q.E.D., Claim).

Since X was arbitrary, $\mathcal{S}$ is a splitting family on κ. However, the size of $\mathcal{S}$ is clearly at most κ. Therefore we have $s(\kappa) \leq \kappa$, a contradiction. (Q.E.D., Lemma 1).

Lemma 2. *If κ is weakly compact, then $\kappa^+ \leq s(\kappa)$.*

Proof of Lemma 2. Underlying idea of our proof is the principle of successive division. Suppose that κ is weakly compact. Assume for a contradiction that $\mathcal{S} = \{S_\alpha : \alpha < \kappa\}$ is a splitting family on κ. For each α, Define S_α^0 and S_α^1 as follows. $S_\alpha^0 = S_\alpha$, $S_\alpha^1 = \kappa \setminus S_\alpha$. Moreover, we define T as follows. $T = \{f \in {}^{<\kappa}2 : |\kappa \cap \bigcap \{S_\alpha^{f(\alpha)} : \alpha \in \mathrm{dom} f\}| = \kappa\}$. Since κ is strongly inaccessible, each level of $(T, \subseteq)$ has size less than κ, and $(T, \subseteq)$ has height κ. Hence, by our assumption that κ is weakly compact, $(T, \subseteq)$ has a branch of size κ. That is, there exists a function $g : \kappa \to 2$ such that for all $\alpha < \kappa$, the initial segment $g \restriction \alpha$ belongs to T.

Case 1. The case where the sequence $\langle \bigcap \{S_\alpha^{g(\alpha)} : \alpha < \xi\} : 0 < \xi < \kappa \rangle$ is eventually constant. In other words, there exists a set $X \in [\kappa]^\kappa$ and an ordinal number $\beta < \kappa$ such that for any ordinal number α, if $\beta \leq \alpha < \kappa$ then $S_\alpha^{g(\alpha)} = X$. Then, for any $\alpha < \kappa$, we have $X \subseteq S_\alpha^{g(\alpha)}$

Case 2. Otherwise. Then, by a diagonal argument, we can show that there exists a set $X \in [\kappa]^\kappa$ such that for any $\alpha < \kappa$, $X \setminus S_\alpha^{g(\alpha)}$ has size less than κ.

In either cases, there exists no $\alpha < \kappa$ such that $|X \cap S_\alpha| = |X \setminus S_\alpha| = \kappa$. Hence $\mathcal{S}$ is not a splitting family on κ, a contradiction. (Q.E.D., Lemma 2).

Now Theorem 1 follows from Lemmas 1 and 2. Q.E.D.

The following presents a lower bound of the consistency strength of the assertion (II′).

Proposition 2. *If we have $\kappa^{++} \leq s(\kappa)$ for some uncountable regular cardinal κ, then there exists an inner model with a measurable cardinal.*

Proof. Assume that κ is an uncountable regular cardinal such that $\kappa^{++} \leq s(\kappa)$. Let K be the Dodd-Jensen core model. Assume for a contradiction that there exists no inner model with a measurable cardinal. Then, by [1], there exists no non-trivial elementary embedding from K to K. Moreover, K satisfies the generalized continuum hypothesis. Hence, by our assumption of $\kappa^{++} \leq s(\kappa)$, $P(\kappa) \cap K$ is not a splitting family on κ: in other words, there exists an $X \in [\kappa]^\kappa$ such that for any $Y \in P(\kappa) \cap K$, the following assertion fails: $|X \cap Y| = |X \setminus Y| = \kappa$. Therefore, letting U be the set $\{Y \in P(\kappa) \cap K : |X \setminus Y| < \kappa\}$, U is a K-κ-complete ultrafilter. Hence there exists a non-trivial elementary embedding from K to K, a contradiction. Q.E.D.

Kamo [5] found an upper bound of the consistency strength of the assertion (II′). He showed that if κ is 2^κ-supercompact, then there exists a generic extension in which the assertion (II′) holds. Tadatoshi Miyamoto at Nanzan University, in his unpublished work, independently found a similar upper bound.

On the other hand, Zapletal [11] improved Proposition 2. That is, he showed that if we have $\kappa^{++} \leq s(\kappa)$ for some uncountable regular cardinal κ, then there exists an inner model with measurable cardinals of high Mitchell order.

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