Simplified proof of an order preserving operator inequality

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A capital letter means a bounded linear operator on a Hilbert space.

Theorem 1. *If $A \geq B \geq 0$ with $A > 0$, then for $1 \geq q \geq t \geq 0$ and $p \geq q$,*

(1) $A^{q-t+r} \geq \{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{(q-t+r)/[(p-t)s+r]}$

for $s \geq 1$ and $r \geq t$.

We cite the following results to prove a simplified proof of Theorem 1 in [3].

Theorem A [1]. *If $A \geq B \geq 0$, then for each $r \geq 0$*

(A-1) $(A^{r/2}A^pA^{r/2})^{1/q} \geq (A^{r/2}B^pA^{r/2})^{1/q}$

holds for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

The domain drawn for p, q and r in Figure is the best possible one [4] for (A-1). Theorem A implies Löwner-Heinz inequality:

(#) $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for $\alpha \in [0, 1]$.

Lemma [2]. *Let X be a positive invertible and Y be an invertible. For any real number λ,*

$$(YXY^*)^\lambda = YX^{1/2}(X^{1/2}Y^*YX^{1/2})^{\lambda-1}X^{1/2}Y^*.$$

Proof of Theorem 1. First of all, we prove that if $A \geq B \geq 0$ with $A > 0$, then

(*) $A^q \geq \{A^{t/2}(A^{-t/2}B^pA^{-t/2})^sA^{t/2}\}^{q/[(p-t)s+t]}$ for $1 \geq q \geq t \geq 0$, $p \geq q$ and $s \geq 1$.

In case $2 \geq s \geq 1$, as $s-1$, $q/[(p-t)s+t] \in [0, 1]$ and $A^t \geq B^t$ by (#), we have (2) by Lemma and (#)

(2) $B_1 = \{A^{t/2}(A^{-t/2}B^pA^{-t/2})^sA^{t/2}\}^{q/[(p-t)s+t]}$
$= \{B^{p/2}(B^{p/2}A^{-t}B^{p/2})^{s-1}B^{p/2}\}^{q/[(p-t)s+t]}$
$\leq \{B^{p/2}(B^{p/2}B^{-t}B^{p/2})^{s-1}B^{p/2}\}^{q/[(p-t)s+t]} = B^q \leq A^q = A_1$ by (#)

for $1 \geq q \geq t \geq 0$, $p \geq q$ and $2 \geq s \geq 1$. Repeating (2) for $A_1 \geq B_1 \geq 0$, then we have

(3) $A_1^{q_1} \geq \{A_1^{t_1/2}(A_1^{-t_1/2}B_1^{p_1}A^{-t_1/2})^{s_1}A_1^{t_1/2}\}^{q_1/[(p_1-t_1)s_1+t_1]}$

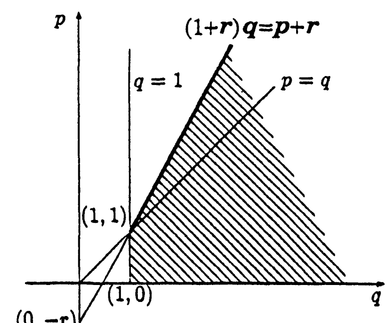
for $1 \geq q_1 \geq t_1 \geq 0$, $p_1 \geq q_1$ and $2 \geq s_1 \geq 1$. Put $1 = q_1 \geq t_1 = t/q \geq 0$ and $p_1 = [(p-t)s+t]/q \geq q_1 = 1$ in (3). Then

(4) $A^q \geq \{A^{t/2}[A^{-t/2}A^{t/2}(A^{-t/2}B^pA^{-t/2})^sA^{t/2}A^{-t/2}]^{s_1}A^{t/2}\}^{q/[(p-t)ss_1+t]}$
$= \{A^{t/2}(A^{-t/2}B^pA^{-t/2})^{ss_1}A^{t/2}\}^{q/[(p-t)ss_1+t]}$ for $1 \geq q \geq t \geq 0$, $p \geq q$ and $4 \geq ss_1 \geq 1$.

Repeating this process from (2) to (4), we obtain (*) for $1 \geq q \geq t \geq 0$, $p \geq q$ and *any* $s \geq 1$. Put $A_2 = A^q$ and $B_2 = \{A^{t/2}(A^{-t/2}B^pA^{-t/2})^sA^{t/2}\}^{q/[(p-t)s+t]}$ in (*). Then $A_2 \geq B_2 \geq 0$ by (*), so by Theorem A we have

(5) $A_2^{1+r_2} \geq (A_2^{r_2/2}B_2^{p_2}A_2^{r_2/2})^{(1+r_2)/(p_2+r_2)}$ for $p_2 \geq 1$ and $r_2 \geq 0$.

We have only to put $r_2 = (r-t)/q \geq 0$ and $p_2 = [(p-t)s+t]/q \geq 1$ in (5) to obtain (1).



Figure

References

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