Power series and asymptotic series associated with the Lerch zeta-function

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1. Introduction. Let s be a complex variable, α and λ real parameters with $\alpha > 0$. Let $\Gamma(s)$ be the gamma-function and $(s)_n = \Gamma(s+n)/\Gamma(s)$ for any integer n denote Pochhammer's symbol. The zeta-function

$$(1.1)\quad \phi(\lambda, \alpha, s) = \sum_{n=0}^{\infty} e^{2\pi i n\lambda}(n+\alpha)^{-s} \quad (\mathrm{Re}\, s > 1)$$

was first introduced and studied by Lerch [10] and Lipschitz [11]. For $\lambda \in \boldsymbol{R}\setminus\boldsymbol{Z}$ it is continued to an entire function over the s-plane, while if $\lambda \in \boldsymbol{Z}$ it reduces to the Hurwitz zeta-function $\zeta(s, \alpha)$. Note that $\zeta(s, 1) = \zeta(s)$ is the Riemann zeta-function.

It is the main aim of the present paper to study power series and asymptotic series for the Lerch zeta-function $\phi(\lambda, \alpha, s)$ in the second parameter (see (1.6) and (2.2) below), based on Mellin-Barnes type of integral formulae. Two applications of our main result will also be presented. For that purpose we extend the domain of the second parameter as follows. Let ω be a real number fixed arbitrarily with $-\pi/2 < \omega < \pi/2$, and S_ω denote the sectorial domain $-\pi/2 + \omega < \arg z < \pi/2 + \omega$. First for any parameter z in S_0, the analytic continuation of $\phi(\lambda, z, s)$ over the s-plane is given by the formula

$$(1.2)\quad \phi(\lambda, z, s) = \frac{1}{\Gamma(s)(e^{2\pi i s-1})}\int_{\mathscr{C}} \frac{e^{-zw}w^{s-1}}{1-e^{2\pi i\lambda - w}}\,dw,$$

where $\mathscr{C}$ is the contour which starts from infinity, proceeds along the real axis to a small positive δ, rounds the origin counter-clockwise, and returns to infinity along the real axis; $\arg w$ varies from 0 to 2π round $\mathscr{C}$. The expression on the right-hand side of (1.2) shows that $\phi(\lambda, z, s)$ is also an analytic function of z in S_0. Next if z is in the intersection of S_0 and S_ω, then the contour $\mathscr{C}$ can be rotated around the origin by an angle $-\omega$ without altering the value of the integral. The resulting formula provides the analytic continuation of $\phi(\lambda, z, s)$ over the s-plane, for any z in S_ω. This operation shows that the domain of z in $\phi(\lambda, z, s)$ can be extended to the whole sector $|\arg z| < \pi$. When $0 < \lambda \le 1$ and $0 < \alpha \le 1$, it follows from (1.2) the functional equation (cf. Erdélyi *et al.* [6], p. 26 and p. 29])

$$(1.3)\quad \phi(\lambda, \alpha, s) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\Big\{e^{\frac{1}{2}\pi i(1-s)}\sum_{l=0}^{\infty} e^{-2\pi i\alpha(l+\lambda)}(l+\lambda)^{s-1} + e^{\frac{1}{2}\pi i(s-1)}\sum_{l=0}^{\infty}{}' e^{2\pi i\alpha(l+1-\lambda)}(l+1-\lambda)^{s-1}\Big\} \quad (\mathrm{Re}\, s < 0),$$

where the prime on the latter summation symbol indicates that the term corresponding to $l = 0$ is to be omitted if $\lambda = 1$.

The present investigation was motivated by the following results (1.4), (1.5) and (1.6), for which we give a brief overview before starting our discussion. It is classically known that the asymptotic expansion

$$(1.4)\quad \sum_{n=1}^{N} n^{-s} \sim \zeta(s) - \frac{N^{1-s}}{s-1} + \frac{1}{2}N^{-s} - \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}(s)_{2k-1}N^{-s-2k+1},$$

as $N \to +\infty$, holds for all $s \ne 1$, where B_k $(k \ge 0)$ is the k-th Bernoulli number. Berndt [4, p. 150] attributed this formula to Ramanujan. Next let $\Psi(a, c; z)$ denote the solution of Kummer's confluent hypergeometric differential equation $zu'' + (c-z)u' - au = 0$ satisfying $\Psi(a, c; z)$

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$\sim z^{-a}$ for $\mathrm{Re}\, a > 0$, as $z \to \infty$ through $|\arg z| < 3\pi/2$ (cf. [6, p. 248 and p. 255]). Ueno and Nishizawa [13] recently introduced, in conjunction with quantum groups, a q-analogue of the Hurwitz zeta-function $\zeta(s, z : q)$, and studied its analytic nature. They proved the formula

$$(1.5)\quad \zeta(s, z) = \frac{z^{1-s}}{s-1} + \frac{1}{2}z^{-s} + \frac{sz^{-s}}{2\pi i}\sum_{l=1}^{\infty} l^{-1} \times \{\Psi(1, 1-s; -2\pi ilz) - \Psi(1, 1-s; 2\pi ilz)\}$$

for $|\arg z| < \pi/2$ and any $s \neq 1, 2, \ldots$, which provides a basis for their derivation of various properties of $\zeta(s, z : q)$. Lastly the power series expansion

$$(1.6)\quad \phi(\lambda, \alpha + z, s) = \sum_{k=0}^{\infty} \frac{(-1)^k (s)_k}{k!} \phi(\lambda, \alpha, s+k) z^k \quad (|z| < \alpha)$$

was established and studied by Klusch [9], who further gave various interesting applications of (1.6).

The formulae (1.4) and (1.5) can in fact be viewed as special cases of a more general asymptotic series for $\zeta(s, \alpha + z)$ in the descending order of z. We shall present this asymptotic series in a more general form (see (2.2) of Theorem 1). Furthermore, a unified treatment of the formulae (1.6) and (2.2) is possible, based on the use of the Mellin-Barnes type of integral (2.6) below. This aspect will be discussed after the proof of Theorem 1.

In the next section we shall state our main result and give a sketch of the proof. Two applications of our main result will be presented in the last section. The detailed version of the proofs and further investigations will appear in forthcoming papers.

2. Main result. Let x and y be complex variables. Apostol [1] introduced the sequence of functions $B_k(x, y)$ $(k \geq 0)$ defined by the Taylor series expansion

$$(2.1)\quad \frac{ze^{xz}}{ye^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(x, y)}{k!} z^k$$

near $z = 0$. The function $B_k(x, y)$, which coincides with the usual Bernoulli polynomial $B_k(x)$ if $y = 1$, is a polynomial in x of degree at most k with coefficients in $\boldsymbol{Q}(y)$.

Our main result can be stated as

Theorem 1. *Let $\varepsilon(\lambda)$ be equal to 0 or 1 according as $\lambda \notin \boldsymbol{Z}$ or $\lambda \in \boldsymbol{Z}$. Then for any integer $K \geq 0$ and any parameter z in the sector $|\arg z| < \pi$, the formula*

$$(2.2)\quad \phi(\lambda, \alpha + z, s) = \frac{\varepsilon(\lambda)}{s-1} z^{1-s} + \sum_{k=0}^{K-1} \frac{(-1)^{k+1}}{(k+1)!} B_{k+1}(\alpha, e^{2\pi i\lambda})(s)_k z^{-s-k} + \rho_K(z; \lambda, \alpha, s)$$

holds in the region $\mathrm{Re}\, s > -K$, where $B_k(\alpha, e^{2\pi i\lambda})$ $(k \geq 0)$ is defined by (2.1). *Here $\rho_K(z; \lambda, \alpha, s)$ is the remainder term satisfying the estimate*

$$(2.3)\quad \rho_K(z; \lambda, \alpha, s) = O(|z|^{-\mathrm{Re}\, s - K}),$$

as $z \to \infty$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$, where the O-constant depends on K, s, λ, α and δ, In particular when $0 < \lambda \leq 1$, $0 < \alpha \leq 1$ and $K \geq 1$ the expression

$$(2.4)\quad \rho_K(z; \lambda, \alpha, s) = \frac{(s)_K z^{1-s-K}}{(2\pi i)^K} \times \Big\{ \sum_{l=0}^{\infty} e^{-2\pi i\alpha(l+\lambda)} (l+\lambda)^{-K} \Psi(1, 2-K-s; 2\pi(l+\lambda) z e^{-\frac{1}{2}\pi i}) + (-1)^K \sum_{l=0}^{\infty}{}' e^{2\pi i\alpha(l+1-\lambda)} (l+1-\lambda)^{-K} \times \Psi(1, 2-K-s; 2\pi(l+1-\lambda) z e^{\frac{1}{2}\pi i}) \Big\}$$

holds for $|\arg z| < \pi$, in the region $\mathrm{Re}\, s > -K$, where the prime has the same meaning as in (1.3).

Remark. When $\lambda \in \boldsymbol{Z}$, the case $s = 1$ of Theorem 1 remains valid if it is regarded as the limiting case $s \to 1$.

Sketch of the proof. Suppose first that Re $s > 1$. Then

$$(2.5)\quad \phi(\lambda, \alpha + z, s) = \sum_{n=0}^{\infty} e^{2\pi in\lambda} (n + \alpha + z)^{-s}$$

for $\alpha > 0$ and any z with $|\arg z| < \pi$. A key to the following derivation is the Mellin transform

$$(n + \alpha + z)^{-s} = \frac{1}{2\pi i} \int_{(b)} \frac{\Gamma(-w)\Gamma(s+w)}{\Gamma(s)} (n+\alpha)^{-s-w} z^w dw,$$

where b is a constant fixed with $1 - \mathrm{Re}\, s < b < 0$, and (b) denotes the vertical straight line from $b - i\infty$ to $b + i\infty$. Substituting this into each term on the right-hand side of (2.5), and changing the order of summation and integration, we obtain

$$(2.6)\quad \phi(\lambda, \alpha + z, s) = \frac{1}{2\pi i} \int_{(b)} \frac{\Gamma(-w)\Gamma(s+w)}{\Gamma(s)} \phi(\lambda, \alpha, s+w) z^w dw.$$

Let K be any nonnegative integer, and b_K a constant fixed with $-\mathrm{Re}\, s - K < b_K < -\mathrm{Re}\, s - K + 1$. We move the path of integration in (2.6) from (b) to (b_K), passing over the poles at $w = 1 - s - k$ $(k = 0, 1, \ldots, K)$. Noting the identities $\phi(\lambda, \alpha, -k) = -B_{k+1}(\alpha, e^{2\pi i\lambda})/(k+1)$ for $k = 0, 1, \ldots$ (cf. [1, p. 164]), and the fact $|z^w| = |z|^{\mathrm{Re}\, w} e^{-\arg z\, \mathrm{Im}\, w}$, we can prove the asymptotic series (2.2), together with the error estimate (2.3). The expression (2.4) follows by applying the functional equation (1.3) and a Mellin-Barnes formula for $\Psi(a, c; z)$ (cf. [6, p. 256]). □

We now return to the consideration of the results (1.4)-(1.6). Theorem 1 with $\lambda = 1$, $\alpha = 1$, $z = N$ and $K = +\infty$ implies (1.4), since the relation $\sum_{n=1}^{N} n^{-s} = \zeta(s) - \zeta(s, N+1)$ holds. Also the formula (1.5) is a special case $\lambda = 1$, $\alpha = 1$ and $K = 1$ of Theorem 1, since $\zeta(s, z) = z^{-s} + \zeta(s, 1+z)$ holds. Moreover, the formula (1.6) can be derived by moving the path of integration in (2.6) to the right, and collecting the residues of the poles at $w = k$ $(k = 0, 1, \ldots)$. The special case $\lambda = 1$ and $\alpha = 1$ of this argument has been given by the author [7][8].

3. Application. In this section we present two applications of Theorem 1.

For the first application we recall the well-known relation

$$\log\frac{\Gamma(z)}{\sqrt{2\pi}} = \frac{\partial}{\partial s}\zeta(s, z)\Big|_{s=0}$$

for $|\arg z| < \pi$ (cf. [6, p. 26]). To pursue this direction further, Deninger [5] introduced and studied the function

$$R(z) = -\frac{\partial^2}{\partial s^2}\zeta(s, z)\Big|_{s=0}$$

for $|\arg z| < \pi$. This led him to important arithmetical applications which include closed evaluations of $L(1, \chi)$ and $L'(1, \chi)$ ($L(s, \chi)$ denotes the Dirichlet L-function) and the analogue of Chowla and Selberg's formula for real quadratic fields. Once and twice differentiations of both sides of our formula (2.2) imply

Corollary 1. *For any integer $K \geq 1$ and any fixed $\alpha > 0$, the formula*

$$\log\frac{\Gamma(z+\alpha)}{\sqrt{2\pi}} = \left(z + \alpha - \frac{1}{2}\right)\log z - z + \sum_{k=1}^{K-1}\frac{(-1)^{k+1}B_{k+1}(\alpha)}{k(k+1)}z^{-k} + O(|z|^{-K})$$

holds, as $z \to \infty$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$, where the O-constant depends on K, α and δ.

This generalized Stirling's formula was originally established by Barnes [2, p. 121] in a slightly different manner.

Corollary 2. *For any integer $K \geq 1$ and any fixed $\alpha > 0$, the formula*

$$R(z+\alpha) = \left(z + \alpha - \frac{1}{2}\right)\log^2 z - 2z\log z + 2z + \sum_{k=1}^{K-1}\frac{(-1)^{k+1}B_{k+1}(\alpha)}{k(k+1)}z^{-k}\left(\log z - \sum_{h=1}^{k-1}\frac{1}{h}\right) + O(|z^{-K}\log z|)$$

holds, as $z \to \infty$ through $|\arg z| < \pi - \delta$ with any small $\delta > 0$, where the O-constant depends on K, α and δ.

This gives a generalization of Deninger's asymptotic series for $R(z)$ (see [5, p. 178]), since the relation $R(z+1) = \log^2 z + R(z)$ holds.

We proceed to the second application. Let u and v be complex variables and ω a positive real parameter. Matsumoto [12] introduced a double zeta-function of the form

$$(3.1)\quad \tilde{\zeta}_2(u, v; \alpha, \omega) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(m+\alpha)^{-u}(m+\alpha+n\omega)^{-v} \quad (\mathrm{Re}\, u > 1,\ \mathrm{Re}\, v > 1),$$

for the purpose of obtaining a better understanding of classical Barnes' double zeta-function $\zeta_2(v; \alpha, \omega)$ (cf. [3]), which is connected with (3.1) by the relation $\zeta_2(v; \alpha, \omega) = \zeta(v, \alpha) + \tilde{\zeta}_2(0, v; \alpha, \omega)$ for $\mathrm{Re}\, v > 2$. He derived an asymptotic expansion for $\tilde{\zeta}_2(u, v; \alpha, \omega)$ in the descending order of ω, which led him to obtain asymptotic series for Barnes' double zeta, double gamma, and Hecke L-functions. Let μ, ν and β be real parameters with $\beta > 0$. Suggested by (3.1), we introduce here a double Lerch zeta-function of the form

$$\phi_2(\mu, \nu, \alpha, \beta; u, v) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}e^{2\pi i(m\mu+n\nu)} \times (m+\alpha)^{-u}(m+n+\alpha+\beta)^{-v} \quad (\mathrm{Re}\, u > 1,\ \mathrm{Re}\, v > 1).$$

We transform this by substituting

$$(m+\alpha)^{-u}(m+n+\alpha+\beta)^{-v} = \frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(-s)\Gamma(v+s)}{\Gamma(v)}(m+\alpha)^{-u-v-s}(n+\beta)^{s}ds$$

(c is a constant fixed with $-\mathrm{Re}\, v < c < -1$) into each term of the double series. Changing the order of summation and integration, we obtain

$$\phi_2(\mu, \nu, \alpha, \beta; u, v) = \frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(-s)\Gamma(v+s)}{\Gamma(v)} \times \phi(\mu, \alpha, u+v+s)\phi(\nu, \beta, -s)ds.$$

Applying Theorem 1 to the inner Lerch zeta-functions $\phi(\mu, \alpha, u+v+s)$ and $\phi(\nu, \beta, -s)$ in this integral expression, we can prove

Theorem 2. *For any integer $K \geq 0$, any positive α and β, and any real μ and ν, the formula*

(3.2) $\phi_2(\mu, \nu, \alpha + x, \beta ; u, v)$

$$= x^{1-u}\int_1^{\infty} \tau^{-u}\phi(\nu, \beta + \tau x, v)d\tau$$

$$- \sum_{k=0}^{K-1} \frac{B_{k+1}(\alpha, e^{2\pi i\mu})}{(k+1)!} x^{-u-k} \frac{\partial^k}{\partial\tau^k}\{\tau^{-u}\phi(\nu, \beta + \tau x, v)\}\Big|_{\tau=1}$$

$$+ O(x^{1-\mathrm{Re}(u+v)-K})$$

holds, as $x \to +\infty$, *in the region* $\mathrm{Re}\, u > 1$ *and* $\mathrm{Re}\, v > 1$, *where the O-constant depends on* u, v, μ, ν, α *and* β.

Theorem 3. *For any integer* $K \geq 0$, *any positive* α *and* β, *and any real* μ *and* ν, *the formula*

(3.3) $\phi_2(\mu, \nu, \alpha, \beta + y ; u, v)$

$$= \frac{\varepsilon(\lambda)}{v-1} \sum_{m=0}^{\infty} e^{2\pi i m\mu}(m+\alpha)^{-u}(m+\alpha+y)^{1-v}$$

$$- \sum_{k=0}^{K-1} \frac{(-1)^k B_{k+1}(\beta, e^{2\pi i\nu})}{(k+1)!}(v)_k \sum_{m=0}^{\infty} e^{2\pi i m\mu}$$

$$\times (m+\alpha)^{-u}(m+\alpha+y)^{-v-k}$$

$$+ O(y^{-\mathrm{Re} v - K})$$

holds, as $y \to +\infty$, *in the region* $\mathrm{Re}\, u > 1$ *and* $\mathrm{Re}\, v > 1$, *where the O-constant depends on* u, v, μ, ν, α *and* β.

Noting that

$$x^{-u-k} \frac{\partial^k}{\partial\tau^k}\{\tau^{-u}\phi(\nu, \beta + \tau x, v)\}\Big|_{\tau=0}$$

$$= \varepsilon(\lambda)\frac{(-1)^k(u+v-1)_k}{v-1}x^{1-u-v-k}$$

$$+ O(x^{-\mathrm{Re}(u+v)-k}),$$

and

$$\sum_{m=0}^{\infty} e^{2\pi i m\mu}(m+\alpha)^{-u}(m+\alpha+y)^{-v-k}$$

$$= \phi(\mu, \alpha, u)y^{-v-k} + O(y^{-\mathrm{Re} v - k - 1}),$$

as $x \to +\infty$ and $y \to +\infty$, for $k = 0, 1, \ldots$, we see that the formulae (3.2) and (3.3) actually give asymptotic expansions in the descending order of x and y, respectively.

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