On Bougerol and Dufresne's identities for exponential Brownian functionals

By Hiroyuki MATSUMOTO*) and Marc YOR**)

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1. Introduction. Let $B = \{B_t\}_{t \geqq 0}$ be a one-dimensional standard Brownian motion starting from 0. To $\{X_t^{(\nu)} = B_t + \nu t\}_{t \geqq 0}$, a Brownian motion with constant drift ν, we associate the exponential additive functional

$$A_t^{(\nu)} = A_t^{(\nu)}(B) = \int_0^t \exp(2(B_s + \nu s))ds, \; t \geqq 0.$$

This Wiener functional plays an important role in a number of domains, mathematical finance (Yor [13], Leblanc [11]), diffusion processes in random environments (Comtet-Monthus [4], Comtet-Monthus-Yor [5], Kawazu-Tanaka [10]), probabilistic studies related to Laplacians on hyperbolic spaces (Gruet [8], Ikeda-Matsumoto [9]) and so on. The readers can find more related topics and references in [16].

We denote A_t for $A_t^{(0)}$. The joint law of (A_t, B_t) is fairly complicated (Yor [13]), although it is quite tractable. But Bougerol's identity ([3]),

(1.1) $\sinh(B_t) \overset{\text{(law)}}{=} \gamma_{A_t}$ for any fixed $t > 0$

for another Brownian motion $\{\gamma_s\}_{s \geqq 0}$ independent of B, makes it easy to calculate the Mellin transform of the probability law of A_t. A simple proof of (1.1), as a consequence of Itô's formula, has been provided in Alili-Dufresne-Yor [1] who have shown the identity in law of the processes

$$\{\exp(B_t) \int_0^t \exp(-B_s) d\gamma_s\}_{t \geqq 0} \text{ and } \{\sinh(B_t)\}_{t \geqq 0}.$$

Another approach to the joint probability law of (A_t, B_t) is found in Alili-Gruet [2] and Ikeda-Matsumoto [9]. These authors have shown independently the following formula. Let ϕ be a function defined by

(1.2) $\phi(x, z) = \sqrt{2}\, e^{x/2} (\cosh z - \cosh x)^{1/2}, \; z \geqq |x|.$

Then it holds that

(1.3) $$E[\exp(-u^2 A_t/2) \,|\, B_t = x] \frac{1}{\sqrt{2\pi t}} \exp(-x^2/2t) = \int_{|x|}^{\infty} \frac{z}{\sqrt{2\pi t^3}} \exp(-z^2/2t) J_0(u\phi(x, z)) dz,$$

for every $t > 0$, where $E[\;|\;]$ is the conditional expectation with respect to the Wiener measure and J_0 is the Bessel function of the first kind of order 0.

Moreover recall that $J_0(\xi)$ is the characteristic function of a symmetrized arcsine random variable $2Z - 1$, where Z is a usual arcsine variable whose probability density is $(\pi \cdot \sqrt{z(1-z)})^{-1}$, $0 < z < 1$. Then, following [2], we can show

(1.4) $(\gamma_{A_t}, B_t) \overset{\text{(law)}}{=} ((2Z - 1)\phi(B_t, \sqrt{R_t^2 + B_t^2}), B_t)$

as a consequence of (1.3), where $R = \{R_t\}_{t \geqq 0}$ is a two-dimensional Bessel process starting from 0 independent of B and Z. We will also see below that (1.4) is equivalent to

$$(\gamma_{A_t}, B_t) \overset{\text{(law)}}{=} ((2Z - 1)\phi(B_t, |B_t| + L_t), B_t)$$

for the local time L_t of B at 0. See Lemmas 1 and 2 in Section 3 for details.

The origin of the present note is the following integral moments formula which has been obtained by Ikeda-Matsumoto [9] by using a result in Yor [13]: for every positive integer n and every $x \in \mathbf{R}$, it holds that

(1.5) $$E[(A_t)^n | B_t = x] \frac{1}{\sqrt{2\pi t}} \exp(-x^2/2t) = \frac{\exp(nx)}{\sqrt{2\pi t^3}\, n!} \int_{|x|}^{\infty} b \exp(-b^2/2t)(\cosh b - \cosh x)^n db.$$

Recall that, letting $\mathbf{e}$ be a standard exponential random variable whose density function is $\exp(-x)$, $x \geqq 0$, we have $E[\mathbf{e}^n] = n!$ and

$$P(\sqrt{2\mathbf{e}t} > b | \sqrt{2\mathbf{e}t} > |a|) = \exp(-a^2/2t) \int_b^{\infty} \frac{c}{t} \exp(-c^2/2t) dc, \; b > |a|.$$

Then (1.5) is equivalent to

(1.6) $$E[(e^{-a}\mathbf{e}A_t)^n | B_t = a] = E[(\cosh\sqrt{2\mathbf{e}t} - \cosh a)^n | \sqrt{2\mathbf{e}t} > |a|],$$

where $\mathbf{e}$ is assumed to be independent of B. Although Carleman's sufficient condition for the

*) School of Informatics and Sciences, Nagoya University, Chikusa-ku, Nagoya 464-8601.

**) Laboratoire de Probabilitiés, Université Pierre et Marie Curie, 4, Place Jussieu, Tour 56, F-75252 Paris Cedex 05, France.

unique solvability of the moment problem is not satisfied and we cannot conclude directly from (1.6), we can show that

$$(1.7)\quad e^{-a}\mathbf{e}A_t(b_a^{(t)}) \overset{(\text{law})}{=} \cosh(\sqrt{2\mathbf{e}t}) - \cosh a \text{ given } \sqrt{2\mathbf{e}t} > |a|$$

holds for any $t > 0$ by using the above mentioned formulae (1.3) and (1.4), where $b_a^{(t)}$ is a Brownian bridge from 0 to a with length t. Therefore the probability law of $(\mathbf{e}A_t(B), B_t)$ is of a simple form.

The purpose of this note is to discuss some identities related to (1.7). In particular, letting L_t be the local time at 0 of the Brownian motion B, we will show that

$$(1.8)\quad (\mathbf{e}e^{-B_t}A_t(B), B_t) \overset{(\text{law})}{=} (\cosh(|B_t| + L_t) - \cosh(B_t), B_t)$$

holds for any $t > 0$, which may help to relate the exponential functional A_t to Pitman's representation theorem of the three-dimensional Bessel process as $\{|B_t| + L_t\}_{t \geqq 0}$. (1.7) is a consequence of (1.8) and the explicit form of the probability density of (B_t, L_t) will play an important role in the proof. In the final section we will show some extensions of a recent interesting result of Dufresne [7] and its relation to our result.

2. Results. The following theorems are the main results of this note. We use the same notations as those in the Introduction.

Theorem 2.1. *For any fixed $t > 0$, it holds that*

$$(2.1)\quad (\mathbf{e}e^{-B_t}A_t(B), B_t) \overset{(\text{law})}{=} (\cosh(|B_t| + L_t) - \cosh(B_t), B_t).$$

Theorem 2.2. *For any fixed $t > 0$, the probability law of $e^{-a}\mathbf{e}A_t(b_a^{(t)})$ coincides with the conditional probability law of $\cosh(\sqrt{2\mathbf{e}t}) - \cosh a$ under the condition $\sqrt{2\mathbf{e}t} > |a|$.*

We postpone proofs of these theorems to the next section and give some consequences of them.

Corollary 2.3. *Let $\{Z_s\}_{s \geqq 0}$ be a complex Brownian motion which is independent of B. Then, for any $t > 0$, it holds that*

$$\left(\frac{1}{2}\left|\int_0^t \exp(B_s - (B_t/2))dZ_s\right|^2, B_t\right) \overset{(\text{law})}{=} (\cosh(|B_t| + L_t) - \cosh(B_t), B_t).$$

Proof. We note that, if $\{\rho_s\}_{s \geqq 0}$ is a real-valued process which is independent of $\{Z_s\}_{s \geqq 0}$, then we have

$$\left(\frac{1}{2}\left|\int_0^t \rho_s dZ_s\right|^2, \rho_t\right) \overset{(\text{law})}{=} \left(\mathbf{e}\int_0^t \rho_s^2 ds, \rho_t\right)$$

for each $t > 0$, where $\mathbf{e}$ is independent of $\{\rho_s\}$. The result now follows immediately from Theorem 2.1. □

Corollary 2.4. *Let $g_t = \sup\{u < t\,;\, B_u = 0\}$ be the last visit to 0 up to time t. Then, for any $t > 0$, it holds that*

$$(2.2)\quad \left(\left|\int_0^{g_t} \exp(B_s)dZ_s\right|, g_t\right) \overset{(\text{law})}{=} (2\sinh(|B_t|/2), t - g_t).$$

Proof. Under the condition $g_t = u$, we have $\{B_s\}_{0 \leqq s \leqq g_t} \overset{(\text{law})}{=} \{\sqrt{u}\, b_0^{(1)}(s/u)\}_{0 \leqq s \leqq u}$. Therefore we get by Theorem 2.2

$$\cosh(\sqrt{2\mathbf{e}u}) - 1 \overset{(\text{law})}{=} \mathbf{e}\int_0^u \exp(2B_s)ds \overset{(\text{law})}{=} \frac{1}{2}\left|\int_0^u \exp(B_s)dZ_s\right|^2.$$

Therefore we obtain, without conditioning:

$$(2.3)\quad \left(\left|\int_0^{g_t} \exp(B_s)dZ_s\right|, g_t\right) \overset{(\text{law})}{=} \left(2\sinh\frac{\sqrt{2g_t\mathbf{e}}}{2}, g_t\right).$$

On the other hand we have

$$|B_t| = \sqrt{t - g_t}\, m_t^{(t)},$$

where $\{m_s^{(t)}\}_{0 \leqq s \leqq t}$ is the Brownian meander defined by

$$m_s^{(t)} = \frac{1}{\sqrt{t - g_t}} |B_{g_t + s(t - g_t)/t}|.$$

It is known (cf. [12]) that $\{m_s^{(t)}\}$ is independent of $t - g_t$ and $m_t^{(t)}$ is identical in law with $\sqrt{2\mathbf{e}}$. Therefore, under the condition $g_t = u$, we get

$$|B_t| \overset{(\text{law})}{=} \sqrt{2(t-u)\mathbf{e}}.$$

Then, noting $g_t \overset{(\text{law})}{=} t - g_t$, we obtain

$$(\sinh(|B_t|/2), t - g_t) \overset{(\text{law})}{=} \left(\sinh\frac{\sqrt{2(t - g_t)\mathbf{e}}}{2}, t - g_t\right) \overset{(\text{law})}{=} \left(\sinh\frac{\sqrt{2g_t\mathbf{e}}}{2}, g_t\right).$$

Combining this with (2.3), we get (2.2). □

Corollary 2.5. *For any $t > 0$, it holds that*

$$\left|\int_0^t \exp(b_0^{(t)}(s))dZ_s\right| \overset{(\text{law})}{=} 2\sinh(L_t(b_0^{(t)})/2).$$

The proof now follows easily by the above mentioned results and is omitted.

3. Proofs of Theorems 2.1 and 2.2. In the following we let $B = \{B_s\}_{s \geqq 0}$, $\gamma = \{\gamma_s\}_{s \geqq 0}$ be one-dimensional standard Brownian motions starting from 0, $R = \{R_s\}_{s \geqq 0}$ a two-dimensional Bessel process starting from 0, Z an arcsine random variable and $\mathbf{e}$ a standard exponential random variable which are defined on a probability space (Ω, P) and are independent of each other. In order to give proofs of the theorems, we prepare two lemmas. Although the first one is due to Alili-Gruet [2], we give a proof for completeness.

Lemma 1. *Let ϕ be the function defined by (1.2). Then, for any $t > 0$, it holds that*

$$(3.1)\quad (\gamma_{A_t}, B_t) \overset{\text{(law)}}{=} ((2Z-1)\phi(B_t, \sqrt{R_t^2+B_t^2}), B_t).$$

Proof. We have only to show that

$$(3.2)\quad E[\exp(\sqrt{-1}u\gamma_{A_t})|B_t=a] = E[\exp(\sqrt{-1}u(2Z-1)\phi(B_t, \sqrt{R_t^2+B_t^2}))|B_t=a]$$

holds for any $u \in \mathbf{R}$ and $a \in \mathbf{R}$, where $E[\,|\,]$ is the conditional expectation with respect to P.

For the left hand side of (3.2), we use (1.3). Then we obtain

$$E[\exp(\sqrt{-1}u\gamma_{A_t})|B_t=a] = E[\exp(-u^2A_t/2)|B_t=a]$$

$$(3.3)\quad = \frac{1}{t}\exp(a^2/2t)\int_{|a|}^{\infty} z\exp(-z^2/2t)J_0(u(\phi(a,z))dz.$$

For the right hand side of (3.2), we recall the integral representation of J_0 and note

$$J_0(\xi) = \frac{1}{\pi}\int_{-1}^{1}\frac{\exp(\sqrt{-1}\xi r)}{\sqrt{1-r^2}}dr$$

$$= \frac{1}{\pi}\int_0^1 \frac{\exp(\sqrt{-1}\xi(2r-1))}{\sqrt{r(1-r)}}dr = E[\exp(\sqrt{-1}(2Z-1)\xi)].$$

Then we get

$$E[\exp(\sqrt{-1}u(2Z-1)\phi(B_t, \sqrt{R_t^2+B_t^2}))|B_t=a] = E[J_0(u\phi(a, \sqrt{R_t^2+a^2}))].$$

Since the probability density of R_t is $t^{-1}z\exp(-z^2/2t)$, we get

$$E[J_0(u\phi(a, \sqrt{R_t^2+a^2}))] = \int_0^{\infty}\frac{z}{t}\exp(-z^2/2t)J_0(u\phi(a, \sqrt{a^2+z^2}))dz = \int_{|a|}^{\infty}\frac{z}{t}\exp(-(z^2-a^2)/2t)J_0(u\phi(a,z))dz$$

and, combining with (3.3), the proof of (3.1) is completed. □

Lemma 2. *Let L_t be the local time of B at 0. Then, for any $t > 0$, it holds that*

$$(3.4)\quad (\sqrt{R_t^2+B_t^2}, B_t) \overset{\text{(law)}}{=} (|B_t| + L_t, B_t).$$

Proof. We have only to prove

$$(3.5)\quad \sqrt{R_t^2+a^2} \overset{\text{(law)}}{=} |a| + L_t(b_a^{(t)})$$

for every $a \in \mathbf{R}$, where $L_t(b_a^{(t)})$ is the local time at 0 of the Brownian bridge $b_a^{(t)}$ from 0 to a with length t. It is easy to show that the probability density of the random variables on both hand sides of (3.5) is given by

$$\frac{1}{t}\exp(-(y^2-a^2)/2t),\ y > |a|.$$

Remark. (3.4) is a variant of Seshadri's identity (cf. [15]).

Now we are in a position to give proofs of Theorems 2.1 and 2.2.

Proof of Theorem 2.1. By Lemma 1, we have

$$(N^2A_t, B_t) \overset{\text{(law)}}{=} (2(2Z-1)^2e^{B_t}(\cosh(\sqrt{R_t^2+B_t^2}) - \cosh(B_t)), B_t),$$

where N is a standard normal random variable independent of B, R, Z and $\mathbf{e}$. Moreover, noting that $\tilde{Z} = (2Z-1)^2$ is again an arcsine variable and $2\tilde{Z}\mathbf{e} \overset{\text{(law)}}{=} N^2$, we obtain

$$(\mathbf{e}A_t, B_t) \overset{\text{(law)}}{=} (e^{B_t}(\cosh(\sqrt{R_t^2+B_t^2}) - \cosh(B_t)), B_t).$$

Therefore, by using Lemma 2, we get (2.1). □

Proof of Theorem 2.2. By Theorem 2.1,we have

$$e^{-a}\mathbf{e}A_t(b_a^{(t)}) \overset{\text{(law)}}{=} \cosh(|a| + L_t(b_a^{(t)})) - \cosh a.$$

Therefore, noting that the conditional probability law of $|a| + L_t$ under the condition $B_t = a$ coincides with that of $\sqrt{2\mathbf{e}t}$ under $\sqrt{2\mathbf{e}t} > |a|$, we get the assertion of Theorem 2.2. □

4. Amplification of Dufresne's identity.

In this final section, we give a relationship between an identity recently obtained by Dufresne [7] and Theorem 2.1. We give only the statements without proofs. The results in this section will be proved and developed elsewhere. We use the same notations as those in the Introduction.

Dufresne's recent result is the following:

Theorem 4.1 ([7]). *For any fixed $t > 0$ and $\mu > 0$, it holds that*

$$(4.1)\quad \frac{1}{A_t^{(-\mu)}} \overset{\text{(law)}}{=} \frac{1}{A_t^{(\mu)}} + \frac{1}{\tilde{A}_\infty^{(-\mu)}},$$

where $\tilde{A}_\infty^{(-\mu)}$ is a copy of $A_\infty^{(-\mu)}$, independent of $A_t^{(\mu)}$.

In fact, we can show that the identity in law (4.1) holds for the stochastic processes involved, by using stability properties of the laws of Bessel processes, both by time reversal (at a last passage time, for transient Bessel processes) and by time inversion. Indeed, we can show the following:

Theorem 4.2. *For every $\mu > 0$, the two-dimensional stochastic processes*

$$\{(e^{X_t^{(-\mu)}}(1 - A_t^{(-\mu)}/A_\infty^{(-\mu)}), e^{-X_t^{(-\mu)}})\}_{t\geqq 0} \text{ and}$$

$$\{(e^{-X_t^{(\mu)}}, e^{-X_t^{(\mu)}}(1 + A_t^{(\mu)}/\tilde{A}_\infty^{(-\mu)}))\}_{t\geqq 0}$$

are identical in low.

Theorem 4.3. (i) *For every $\mu > 0$, the two-dimensional stochastic processes*

$$\{(e^{-X_t^{(-\mu)}} A_t^{(-\mu)}, A_t^{(-\mu)})\}_{t \geqq 0} \text{ and}$$

$$\{(e^{-X_t^{(\mu)}} A_t^{(\mu)}, \tilde{A}_\infty^{(-\mu)} A_t^{(\mu)}/(\tilde{A}_\infty^{(-\mu)} + A_t^{(\mu)}))\}_{t \geqq 0}$$

are identical in law.

(ii) *The random variable* $A_\infty^{(-\mu)}$ *is independent of the stochastic process* $\{e^{-X_t^{(-\mu)}} A_t^{(-\mu)}\}_{t \geqq 0}$.

Note that $(2A_\infty^{(-\mu)})^{-1}$ obeys the *Gamma* (μ) distribution (Dufresne [6], Yor [14]) and, in particular, $(2A_\infty^{(-1)})^{-1}$ is a standard exponential random variable. Therefore, by using Cameron-Martin's Theorem, we can show by Theorem 2.1 that

$$(2\cosh(L_t^{(1)} + |X_t^{(1)}|) - e^{X_t^{(1)}}, e^{X_t^{(1)}}) \overset{\text{(law)}}{=} (e^{-X_t^{(1)}}(1 + A_t^{(1)}/\tilde{A}_\infty^{(-1)}), e^{X_t^{(1)}}),$$

holds for any $t > 0$, where $L_t^{(1)}$ is the local time of $X_t^{(1)}$ at 0.

Combining this with Theorem 4.2, we obtain the following:

Theorem 4.4. *For any fixed* $t > 0$, *the law of* $(2\cosh(L_t^{(1)} + |X_t^{(1)}|) - e^{X_t^{(1)}}, e^{X_t^{(1)}})$ *is symmetric and it is also the common law of the following pairs of random variables*

$$(e^{-X_t^{(1)}}(1 + A_t^{(1)}/\tilde{A}_\infty^{(-1)}), e^{X_t^{(1)}}) \text{ and}$$

$$(e^{-X_t^{(-1)}}, e^{X_t^{(-1)}}(1 - A_t^{(-1)}/A_\infty^{(-1)})).$$

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