The lifted Futaki invariants for Riemann surfaces

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Abstract: It is conjectured that the lifted Futaki invariant of an n-dimensional compact complex manifold vanishes if it admits an Einstein-Kähler metric. If the conjecture holds for $n = 1$, the lifted Futaki invariants for Riemann surfaces must vanish because Riemann surfaces always admit Einstein-Kähler metrics.

In this paper, we prove the vanishing of the lifted Futaki invariants for Riemann surfaces under a certain assumption. Our main result is Theorem 1.3.

Key words: The lifted Futaki invariant; complex manifold; Einstein-Kähler metric; Riemann surface.

1. Introduction and main theorem. Let M be a compact complex manifold, $A(M)$ the complex Lie group consisting of the biholomorphic automorphisms of M and $V(M)$ its Lie algebra consisting of the holomorphic vector fields on M. Then, in [1] (See also [2]), Futaki defined a Lie algebra homomorphism $f : V(M) \longrightarrow \mathbf{C}$, which is called the "Futaki invariant", and showed that $f(X) = 0$ for any $X \in V(M)$ if M admits an Einstein-Kähler metric. In [2], using the Simons character of a certain foliation, Futaki-Morita defined a Lie group homomorphism $F : A(M) \longrightarrow \mathbf{C}/\mathbf{Z}$ (where $\mathbf{C}/\mathbf{Z}$ is the additive group), which is called the "lifted Futaki invariant" (see also [4]). The lifted Futaki invariant F satisfies the condition that $F(\exp X) = f(X) \bmod \mathbf{Z}$ for any $X \in V(M)$. As was shown in [3], F may be non-zero even when $V(M) = \{0\}$.

Now let M be a compact connected Riemann surface of genus σ with any complex structure, $g \in A(M)$ a periodic automorphism of order p and $\Omega(k)$ the fixed point set of g^k $(1 \leq k \leq p-1)$. Then the next theorem is the immediate consequence of Theorem 2.10 in [4].

Theorem 1.1. *Assume that g^k acts on the tangent space T_qM for $q \in \Omega(k)$ via multiplication by $\xi_p^{m_q} \neq 1$ where $\xi_p = e^{2\pi\sqrt{-1}/p}$ and $m_q \in \mathbf{Z}$. Then we have*

$$\mathbf{C}/\mathbf{Z} \ni F(g) = \frac{1}{p}\sum_{k=1}^{p-1}\sum_{q\in\Omega(k)} \xi_p^{k+m_q}\frac{\xi_p^{m_q}-1}{\xi_p^k-1}.$$

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Now suppose that $p = p_1^{n_1}p_2^{n_2}\cdots p_e^{n_e}$ where $p_1, p_2, \ldots, p_e$ are mutually distinct prime numbers and $n_1, n_2, \ldots, n_e$ are natural numbers. Set $g_i = g^{p/p_i^{n_i}}$ for $1 \leq i \leq e$. Then the order of g_i is equal to $p_i^{n_i}$.

Assumption 1.2. *When $\sigma \geq 2$, we assume that there exists a natural number m_i which is prime to p_i such that $g_i^{p_i^r}$ acts on the tangent space T_qM of the fixed point q of $g_i^{p_i^r}$ via multiplication by $\exp(2\pi\sqrt{-1}\, m_i\varepsilon_{q,r}/p_i^{n_i-r})$ for any $1 \leq r \leq n_i - 1$ and any q where $\varepsilon_{q,r}$ is equal to 1 or -1.*

Our main theorem is the next theorem.

Theorem 1.3. *Under the assumption above, the lifted Futaki invariant $F(g)$ vanishes.*

2. Proof of the main theorem. When $\sigma = 0$, $A(M) = A(\mathbf{CP}^1) = PGL(2;\mathbf{C})$ is connected and hence there exists $X \in V(M)$ such that $g = \exp X$. Therefore $F(g) = f(X) = 0$ because M admits an Einstein-Kähler metric. So we assume that $\sigma \geq 1$ hereafter. First assume that $\sigma \geq 2$. Since m_1 is prime to p_1, there exists a natural number ℓ such that $m_1\ell \equiv 1 \pmod{p_1^{n_1}}$. Set $g_* := g_1^\ell$. Then $g_1 = g_*^{m_1}$, the order of g_* is $p_1^{n_1}$ and the fixed point set of g_*^r coincides with that of g_1^r for any r. Let $\Omega_*(k)$ be the fixed point set of g_*^k $(1 \leq k \leq p_1^{n_1} - 1)$. Then it follows from Assumption 1.2 that $g_*^{p_1^r}$ acts on the tangent space T_qM of $q \in \Omega_*(p_1^r)$ via multiplication by $\alpha^{p_1^r\varepsilon_{q,r}} = \beta^{\varepsilon_{q,r}}$ where $\alpha = \exp\left(2\pi\sqrt{-1}/p_1^{n_1}\right)$ and $\beta = \alpha^{p_1^r} = \exp\left(2\pi\sqrt{-1}/p_1^{n_1-r}\right)$.

Since $\Omega_*(p_1^{r-1}) \subset \Omega_*(p_1^r)$, we can define the set S_r consisting of fixed points by

$$S_0 = \Omega_*(1),$$
$$S_r = \Omega_*(p_1^r) \setminus \Omega_*(p_1^{r-1}) \quad (1 \leq r \leq n_1 - 1).$$

Then the following lemmas hold.

Lemma 2.1. (1) *$\Omega_*(p_1^r)$ is the disjoint union of S_0, S_1, ..., S_r.*

(2) *$\Omega_*(tp_1^r) = \Omega_*(p_1^r)$ if t is not a multiple of p_1.*

(3) *Assume that $S_r \neq \phi$. Then we have*

$$S_r \cap \Omega_*(k) \neq \phi \Longleftrightarrow S_r \subset \Omega_*(k) \Longleftrightarrow k = tp_1^r$$

for $1 \leq t \leq p_1^{n_1-r} - 1$.

(4) *The set S_r is invariant under the action of g_*^k for any k.*

(5) *There exist points $\{q_j^r\}_{j=1}^{N(r)}$ such that S_r is the disjoint sum of the points $g_*^i \cdot q_j^r$ for*

$$i = 0, 1, \dots, p_1^r - 1, \ j = 1, 2, \dots, N(r) \quad \text{if } S_r \neq \phi.$$

Proof. (1) This follows immediately from the definition of S_r.

(2) Assume that $k = tp_1^r$ where $0 \leq r \leq n_1 - 1$ and t is not a multiple of p_1. Since tp_1^r is a multiple of p_1^r, it is clear that $\Omega_*(tp_1^r) \supset \Omega_*(p_1^r)$. On the other hand, since the greatest common divisor of t, $p_1^{n_1-r}$ is equal to 1, there exists a natural number λ such that $t\lambda = 1 + \mu p_1^{n_1-r}$ for some natural number μ and hence we have

$$(g_*^{tp_1^r})^\lambda = g_*^{tp_1^r\lambda} = g_*^{p_1^r + \mu p_1^{n_1}} = g_*^{p_1^r}.$$

Therefore, it follows that $\Omega_*(tp_1^r) \subset \Omega_*(p_1^r)$ and hence that $\Omega_*(tp_1^r) = \Omega_*(p_1^r)$.

(3) If $k = tp_1^r$ for $1 \leq t \leq p_1^{n_1-r} - 1$, it follows that $\Omega_*(k) \supset \Omega_*(p_1^r)$ and hence that $\Omega_*(k) \supset S_r$. If $\Omega_*(k) \supset S_r$, it is clear that $\Omega_*(k) \cap S_r \neq \phi$. If k is not a multiple of p_1^r, there exist non-negative integer j and a natural number t such that $0 \leq j < r$, t is prime to p_1 and that $k = tp_1^j$. Therefore it follows from (1) and (2) that $\Omega_*(k)$ is the disjoint union of $S_0, S_1, \dots, S_j$ and hence that $S_r \cap \Omega_*(k) = \phi$.

(4) We have

$$g_*^{p_1^r} \cdot g_*^k \cdot q = g_*^k \cdot g_*^{p_1^r} \cdot q = g_*^k \cdot q$$
$$g_*^{p_1^t} \cdot g_*^k \cdot q = g_*^k \cdot g_*^{p_1^t} \cdot q \neq g_*^k \cdot q \quad \text{if } t < r$$

for any k which implies that $g_*^k \cdot S_r = S_r$ for any r.

(5) Since $\mathbf{Z}/p_1^r\mathbf{Z} = \langle g_* \rangle / \langle g_*^{p_1^r} \rangle$ acts freely on S_r where $\langle g_* \rangle$ denotes the cyclic group generated by g_*, there exist points $\{q_j^r\}_{j=1}^{N(r)}$ in M which represent $S_r/(\mathbf{Z}/p_1^r\mathbf{Z})$. Then S_r is the disjoint sum of the points $g_*^i \cdot q_j^r$ for $i = 0, 1, \dots, p_1^r - 1$, $j = 1, 2, \dots, N(r)$. □

Note that $g_*^{p_1^r}$ acts on the tangent space $T_{g_*^i \cdot q_j^r} M$ via multiplication by $\beta^{\varepsilon_{r,j}}$ for any r, i, j where $\varepsilon_{r,j}$ is equal to 1 or -1 because g_*^i acts isometrically on M.

Lemma 2.2.

$$\sum_{r=0}^{n_1-1} \sum_{j=1}^{N(r)} p_1^r \varepsilon_{r,j} \equiv 0 \pmod{p_1^{n_1}}.$$

Proof. Let $\tilde{M}$ denote the punctured surface defined by $M \setminus (\cup_{r=0}^{n_1-1} S_r)$. Then $\langle g_* \rangle = \mathbf{Z}/p_1^{n_1}\mathbf{Z}$ and $\langle g_*^{p_1^r} \rangle = \mathbf{Z}/p_1^{n_1-r}\mathbf{Z}$ acts freely on $\tilde{M}$ and hence we can define $\mathbf{Z}/p_1^{n_1}\mathbf{Z}$-covering

$$P : \tilde{M} \longrightarrow \bar{M} = \tilde{M}/(\mathbf{Z}/p_1^{n_1}\mathbf{Z})$$

and $\mathbf{Z}/p_1^{n_1-r}\mathbf{Z}$-covering

$$\hat{P} : \tilde{M} \longrightarrow \hat{M} = \tilde{M}/(\mathbf{Z}/p_1^{n_1-r}\mathbf{Z}).$$

Moreover $\langle g_* \rangle / \langle g_*^{p_1^r} \rangle = \mathbf{Z}/p_1^r\mathbf{Z}$ acts freely on $\hat{M}$ and hence we can define $\mathbf{Z}/p_1^r\mathbf{Z}$-covering

$$\bar{P} : \hat{M} \longrightarrow \hat{M}/(\mathbf{Z}/p_1^r\mathbf{Z}) = \bar{M}.$$

Therefore we have an exact sequence

$$\pi_1(\tilde{M}) \longrightarrow \pi_1(\bar{M}) \xrightarrow{\psi} \pi_0(\mathbf{Z}/p_1^{n_1}\mathbf{Z}) = \mathbf{Z}/p_1^{n_1}\mathbf{Z} \longrightarrow 0.$$

Fix a base point $\tilde{O} \in \tilde{M}$. Let $\hat{O}$ be the point in $\hat{M}$ defined by $\hat{O} = \hat{P}(\tilde{O})$ and O the point in $\bar{M}$ defined by $O = P(\tilde{O})$. Let $g_*^i \cdot q_j^r$ be any point in S_r. Let P_o denote the projection from M to $M_o := M/\langle g_* \rangle$ and $\gamma_j^{i,r}$ a counterclockwise loop in M_o with respect to the orientation of M_o around $P_o(g_*^i \cdot q_j^r) = P_o(q_j^r)$ which starts at O. Since $\gamma_j^{i,r}$ lifts to a loop $\hat{\gamma}_j^{i,r}$ in $\hat{M}$ which starts at $\hat{O}$ and the loop $\hat{\gamma}_j^{i,r}$ lifts to a curve connecting a point $\tilde{O} \in \tilde{M}$ to $g_*^{p_1^r \varepsilon_{r,j}} \cdot \tilde{O} \in \tilde{M}$ because the automorphism g_*^i commutes with $g_*^{p_1^r}$ and $g_*^{p_1^r}$ acts on the tangent space $T_{g_*^i \cdot q_j^r} M$ via multiplication by $\beta^{\varepsilon_{r,j}}$ for any r, i, j. Since $\bar{P} \circ \hat{P} = P$, it follows that $\psi(\gamma_j^{i,r}) = p_1^r \varepsilon_{r,j} \in \mathbf{Z}/p_1^{n_1}\mathbf{Z}$ for any r, i, j. Since $\mathbf{Z}/p_1^{n_1}\mathbf{Z}$ is Abelian, ψ factors through a homomorphism $H_1(\bar{M}) \longrightarrow \mathbf{Z}/p_1^{n_1}\mathbf{Z}$. On the other hand, $\bar{M}$ is homeomorphic to the punctured surface obtained by removing the points $\{P_o(q_j^r)\}$ in M_o. Let σ_o denote the genus of the punctured surface $\bar{M}$, namely the genus of M_o. If $\sigma_o \geq 1$, $\pi_1(M_o)$ is the free group generated by 1-cells $\{a_i, b_i\}_{i=1}^{\sigma_o}$ with fundamental relation $\prod_{i=1}^{\sigma_o} a_i b_i a_i^{-1} b_i^{-1} = 1$. We can assume that the loops representing $\{a_i, b_i\}$ are contained in $\bar{M}$. Thus, by cutting open $\bar{M}$ along these loops, $\bar{M}$ is shown to be homeomorphic to the

$4\sigma_o$-gon $\triangle_{4\sigma_o}$ punctured at the points corresponding to $\cup_{r=0}^{n_1-1}\cup_{j=1}^{N(r)} P_o(q_j^r)$ which are contained in the interior of $\triangle_{4\sigma_o}$. Then $\sum_{r=0}^{n_1-1}\sum_{j=1}^{N(r)}\gamma_j^{i,r}$ represent a 0-homologous element in $H_1(\bar{M})$ because $\sum_{r=0}^{n_1-1}\sum_{j=1}^{N(r)}\gamma_j^{i,r}$ is homologous to the boundary of $\triangle_{4\sigma_o}$ which represent a 0-homologous element in $H_1(\bar{M})$. Thus we have

$$\mathbf{Z}/p_1^{n_1}\mathbf{Z} \ni 0 = \psi\left(\sum_{r=0}^{n_1-1}\sum_{j=1}^{N(r)}\gamma_j^{i,r}\right) = \sum_{r=0}^{n_1-1}\sum_{j=1}^{N(r)} p_1^r\varepsilon_{r,j}.$$

□

Lemma 2.3. $\mathbf{C}/\mathbf{Z} \ni F(g_*) = 0$.

Proof. Since $g_*^{tp_1^r} = (g_*^{p_1^r})^t$ acts on $T_{q_j^r}M$ via multiplication by $\beta^{\varepsilon_{r,j}}$, it follows from Theorem 1.1, Lemma 2.1 and Lemma 2.2 that

$$\begin{aligned}
&F(g_*) = \ell F(g_1)\\
&= \frac{1}{p_1^{n_1}}\sum_{k=1}^{p_1^{n_1}-1}\sum_{q\in\Omega_*(k)}\alpha^{k+m_q}\frac{\alpha^{m_q}-1}{\alpha^k-1}\\
&= \frac{1}{p_1^{n_1}}\sum_{r=0}^{n_1-1}\sum_{t=1}^{p_1^{n_1-r}-1}\sum_{j=1}^{N(r)}\\
&\qquad\qquad \sum_{i=0}^{p_1^r-1}\alpha^{p_1^r t(1+\varepsilon_{r,j})}\frac{\alpha^{p_1^r t\varepsilon_{r,j}}-1}{\alpha^{p_1^r t}-1}\\
&= \frac{1}{p_1^{n_1}}\sum_{r=0}^{n_1-1}\sum_{j=1}^{N(r)}p_1^r\sum_{t=1}^{p_1^{n_1-r}-1}\beta^{(1+\varepsilon_{r,j})t}\frac{\beta^{t\varepsilon_{r,j}}-1}{\beta^t-1}\\
&\left(\text{since} \sum_{t=1}^{p_1^{n_1-r}-1}\beta^{(1+\varepsilon_{r,j})t}\frac{\beta^{t\varepsilon_{r,j}}-1}{\beta^t-1} \equiv -\varepsilon_{r,j} \pmod{p_1^{n_1-r}}\right)\\
&= -\frac{1}{p_1^{n_1}}\sum_{r=0}^{n_1-1}\sum_{j=1}^{N(r)}p_1^r\varepsilon_{r,j} = 0 \pmod{\mathbf{Z}}.
\end{aligned}$$

Since ℓ is prime to p_1, $\ell F(g_1) = 0$ implies that $F(g_1) = 0$. □

Now let $\mathrm{ord}(F(g))$ denote the order of $F(g) \in \mathbf{C}/\mathbf{Z}$ defined by

$$\begin{aligned}
&\mathrm{ord}(F(g))\\
&= \min\{n \in \mathbf{N} \mid nF(g) = F(g^n) = 0 \in \mathbf{C}/\mathbf{Z}\},
\end{aligned}$$

which is a divisor of $p = p_1^{n_1}p_2^{n_2}\cdots p_e^{n_e}$. Since $F(g_1) = 0 \in \mathbf{C}/\mathbf{Z}$, $\mathrm{ord}(F(g))$ is a divisor of $p_2^{n_2}\cdots p_e^{n_e}$ and therefore is prime to p_1. The same argument deduces that $\mathrm{ord}(F(g))$ is prime to p_i for $1 \le i \le e$ and that $\mathrm{ord}(F(g)) = 1$. This implies that $F(g) = 0$.

Next we consider the case when $\sigma = 1$. Then the universal covering of M is $\mathbf{C}$ and we may assume that $M = \mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z})$ where $\tau \in \mathbf{C}$ satisfies $1 \le |\tau|$, $0 \le \mathrm{Re}(\tau) \le 1/2$ and $0 < \mathrm{Im}(\tau)$. Moreover there exists a splitting exact sequence

$$1 \longrightarrow A_0(M) \longrightarrow A(M) \longrightarrow H \longrightarrow 1$$

where $A_0(M)$ is the identity component of $A(M)$ and H is the cyclic group. Let h be a generator of H. Then for any $g \in A(M)$ $h^k g$ is contained in $A_0(M)$ for some k and hence there exists $X \in V(M)$ such that $h^k g = \exp X$. Therefore, we have

$$\begin{aligned}
F(g) &= F(h^{-k}\exp X) = -kF(h) + F(\exp X)\\
&= -kF(h) + f(X) = -kF(h) \in \mathbf{C}/\mathbf{Z}
\end{aligned}$$

because M admits an Einstein-Kähler metric. So it suffices to show that $F(h) = 0 \in \mathbf{C}/\mathbf{Z}$.

If $\tau = \exp(2\pi\sqrt{-1}/6)$, H is the cyclic group of order 6 generated by an automorphism h_6 defined by the multiplication by τ. Then we can see that

$$\begin{aligned}
&\Omega_6(1) = \Omega_6(5) = \{0\},\\
&\Omega_6(2) = \Omega_6(4) = \left\{0, \frac{1+\tau}{3}, \frac{2(1+\tau)}{3}\right\},\\
&\Omega_6(3) = \left\{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\right\}
\end{aligned}$$

where $\Omega_6(k)$ denotes the fixed point set of h_6^k. Since h_6 acts on the tangent space of each fixed point via multiplication by τ, it follows from Theorem 1.1 that

$$\begin{aligned}
F(h_6) &= \frac{1}{6}\left(\tau^2\frac{\tau-1}{\tau-1} + 3\tau^4\frac{\tau^2-1}{\tau^2-1} + 4\tau^6\frac{\tau^3-1}{\tau^3-1}\right.\\
&\qquad\left. + 3\tau^8\frac{\tau^4-1}{\tau^4-1} + \tau^{10}\frac{\tau^5-1}{\tau^5-1}\right)\\
&= \frac{1}{6}(4\tau^4 + 4\tau^2 + 4) = 0 \in \mathbf{C}/\mathbf{Z}.
\end{aligned}$$

If $\tau = \exp(2\pi\sqrt{-1}/4)$, H is the cyclic group of order 4 generated by an automorphism h_4 defined by the multiplication by τ. Then we can see that

$$\begin{aligned}
&\Omega(1) = \Omega(3) = \left\{0, \frac{1+\tau}{2}\right\},\\
&\Omega(2) = \left\{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\right\}
\end{aligned}$$

and h_4 acts on the tangent space of each fixed point via multiplication by τ. Therefore it follows from

Theorem 1.1 that

$$F(h_4) = \frac{1}{4}\left(2\tau^2\frac{\tau-1}{\tau-1} + 4\tau^4\frac{\tau^2-1}{\tau^2-1} + 2\tau^6\frac{\tau^3-1}{\tau^3-1}\right)$$
$$= \frac{1}{4}(4\tau^4 + 4\tau^2) = 0 \in \mathbf{C}/\mathbf{Z}.$$

In other cases, H is the cyclic group of order 2 generated by an automorphism h_2 defined by the multiplication by -1. Then we can see that $\Omega(1) = \{0, (1+\tau)/2\}$ and h_2 acts on the tangent space of the fixed point via multiplication by -1. Therefore it follows from Theorem 1.1 that

$$F(h_2) = \frac{1}{2}\left(2(-1)^2\frac{(-1)-1}{(-1)-1}\right) = 0 \in \mathbf{C}/\mathbf{Z}\,.$$

This completes the proof of Theorem 1.3. □

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