The unitary part of class $\mathcal{F}$ contractions

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Abstract: We say that a bounded linear operator T on a Hilbert space $\mathcal{H}$ belongs to the class $\mathcal{F}$ if T satisfies the following Fuglede's property that, for a given isometry W on $\mathcal{H}$, $SW^* = TS$ for some bounded linear operator S on $\mathcal{H}$ always implies $SW = T^*S$. Such class is wider than the class of paranormal contractions, the class of dominant operators and the class $\mathcal{Y}$ which was introduced in [4]. In this paper, we prove that, for the class $\mathcal{F}$ contraction T on $\mathcal{H}$, the positive square root A_{T^*} of the strong limit of T^nT^{*n} is the projection from $\mathcal{H}$ onto $\mathcal{H}_T^{(u)}$ on which the unitary part of T acts.

Key words: contraction; unitary part; hyponormal operators; paranormal operators; dominant operators.

1. Introduction. It is known that, for a **contraction** T (i.e., $\|T\| \leq 1$) on a Hilbert space $\mathcal{H}$,

$$\begin{aligned}\mathcal{H}_T^{(u)} &\stackrel{\text{def}}{=} \{x \in \mathcal{H} \ ; \ \|T^k x\| = \|x\| = \|T^{*k}x\| \\ &\qquad\qquad \text{for all } k = 1, 2, \cdots\} \\ &= \cap_{k=1}^{\infty}\{x \in \mathcal{H} \ ; \ T^{*k}T^k x = x = T^k T^{*k} x\}\end{aligned}$$

is the maximal reducing subspace on which its restriction is unitary and that the projection from $\mathcal{H}$ onto $\mathcal{H}_T^{(u)}$ belongs to the centre of $\mathcal{R}(T)$, where $\mathcal{R}(T)$ is the von Neumann algebra generated by T. The unitary operator $T|_{\mathcal{H}_T^{(u)}}$ is called the unitary part of T.

If T is a contraction, then $\|T^{n+1}x\| \leq \|T^n x\|$ for all $x \in \mathcal{H}$ and the sequence $\{T^{*n}T^n\}$ is monotonically decreasing and hence it converges to a positive contraction ${A_T}^2$ strongly and $T^*{A_T}^2T = {A_T}^2$. By using the unique positive square root A_T of ${A_T}^2$, we can represent $\mathcal{H}_T^{(u)}$ as follows:

$$\begin{aligned}\mathcal{H}_T^{(u)} &= \{x \in \mathcal{H} \ ; \ \|A_T x\| = \|A_{T^*}x\| = \|x\|\} \\ &= \{x \in \mathcal{H} \ ; \ {A_T}^2 x = {A_{T^*}}^2 x = x\} \\ &= \mathcal{N}_{I-A_T} \cap \mathcal{N}_{I-A_{T^*}},\end{aligned}$$

where $\mathcal{N}_B$ denotes the null space of the operator B.

It is clear that $\mathcal{N}_{A_T} = \{x \in \mathcal{H} : A_T x = 0\}$ and

$$\begin{aligned}\mathcal{N}_{I-A_T} &= \{x \in \mathcal{H} : A_T x = x\} \\ &= \{x \in \mathcal{H} : \|T^n x\| = \|x\|, \ n = 1, 2, \cdots\}\end{aligned}$$

are invariant under T and $T|_{\mathcal{N}_{I-A_T}}$ is an isometry and

$$\mathcal{N}_{A_T - {A_T}^2} = \mathcal{N}_{A_T} \oplus \mathcal{N}_{I-A_T}.$$

In [3], C. R. Putnam proved the following.

Proposition. *If T is a* **hyponormal** *(i.e., $T^*T \geq TT^*$) contraction on $\mathcal{H}$, then A_{T^*} is the projection from $\mathcal{H}$ onto $\mathcal{H}_T^{(u)}$.*

This result is generalized in the each case where T is a **paranormal** (i.e., $\|Tx\|^2 \leq \|T^2x\| \, \|x\|$ for all $x \in \mathcal{H}$) contraction by K. Ôkubo [2] and where T is a **dominant** (i.e., $(T - zI)\mathcal{H} \subseteq (T - zI)^*\mathcal{H}$ for all $z \in \sigma(T)$, where $\sigma(T)$ denotes the spectrum of T) contraction by [6] respectively.

In this paper we shall show that Proposition is generalized for the more wide class of contractions defined as follows.

Definition 1. If a bounded linear operator T on $\mathcal{H}$ satisfies the following Fuglede's property that, for a given isometry W on $\mathcal{H}$, $SW^* = TS$ for some bounded linear operator S on $\mathcal{H}$ always implies $SW = T^*S$, then we say that T belongs to the **class** $\mathcal{F}$ and denotes $T \in$ the class $\mathcal{F}$.

It is known that the paranormal contractions and also the dominant operators belong to the class $\mathcal{F}$ by E. Goya and T. Saitô [1] and by [5] respectively.

In [4], we defined the following class of opera-

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tors.

Definition 2. For a bounded linear operator T on $\mathcal{H}$, we say that T belongs to the **class** $\mathcal{Y}_\alpha$ for some $\alpha \geq 1$ if there is a positive number K_α such that

$$|T^*T - TT^*|^\alpha \leq {K_\alpha}^2 (T - zI)^*(T - zI) \quad \text{for all } z \in \mathbf{C},$$

where $|B|$ denotes the absolute value $(B^*B)^{\frac{1}{2}}$ of the operator B and $\mathbf{C}$ denotes the set of all complex numbers. It is known that, for each $\alpha,\ \beta$ such as $1 \leq \alpha < \beta$, $\mathcal{Y}_\alpha \subseteq \mathcal{Y}_\beta$ and we say that the operator in $\mathcal{Y} = \cup_{\alpha \geq 1} \mathcal{Y}_\alpha$ is the **class** $\mathcal{Y}$ operator.

It is also known that the class $\mathcal{Y}$ operators belong to the class $\mathcal{F}$ by [4]. Each class of operators, that is, the class of paranormal operators, the class of dominant operators and the class $\mathcal{Y}$, contains the hyponormal operators but these classes are mutually distinct.

2. Preliminaries. Throughout this section, let T be a contraction on $\mathcal{H}$. Firstly we shall study the general properties of A_T.

Lemma 1. *For any positive integer n, $\|A_T T^n x\| = \|A_T x\| \geq \|T^{*n} A_T x\|$ for all $x \in \mathcal{H}$ and $A_T T^n$ is hyponormal.*

Proof. For any $x \in \mathcal{H}$, we have

$$\begin{aligned}\|A_T T^n x\|^2 &= \langle T^{*n} {A_T}^2 T^n x,\ x\rangle = \langle {A_T}^2 x,\ x\rangle \\ &= \|A_T x\|^2 \geq \|T^{*n} A_T x\|^2. \end{aligned}$$ □

Let $A_T T = V_T A_T$ be the polar decomposition of $A_T T$. Then V_T is a partial isometry and $\mathcal{N}_{V_T} = \mathcal{N}_{A_T}$.

Lemma 2. $[A_T\mathcal{H}]^\sim$ *reduces V_T where "$\sim$" denotes the closure. And hence the restriction $V_T|_{[A_T\mathcal{H}]^\sim}$ is an isometry because $\mathcal{N}_{V_T} = \mathcal{N}_{A_T}$.*

Proof. Since $A_T T = V_T A_T$, $[A_T\mathcal{H}]^\sim$ is invariant under V_T and since $\mathcal{N}_{V_T} = \mathcal{N}_{A_T}$, $\mathcal{N}_{A_T}$ is invariant under V_T. Therefore $[A_T\mathcal{H}]^\sim$ reduces V_T. □

Lemma 3. *A necessary and sufficient condition that A_T is the projection from $\mathcal{H}$ onto $\mathcal{H}_T^{(u)}$ is that $A_T T$ is normal.*

Proof. Assume that $A_T T$ is normal. Since

$A_T T$ is normal

$\Leftrightarrow$ $\|T^* A_T x\| = \|A_T x\|$ for all $x \in \mathcal{H}$ (by Lemma 1)

$\Leftrightarrow$ $TT^* A_T x = A_T x$ for all $x \in \mathcal{H}$ (because $T$is a contraction)

$\Leftrightarrow$ $TT^* A_T = A_T$,

we have

$$T{A_T}^2 = TT^*{A_T}^2 T = {A_T}^2 T$$

and A_T commutes with T. And then

$$T^{*n}T^n {A_T}^2 = T^{*n}{A_T}^2 T^n = {A_T}^2$$

and ${A_T}^4 = {A_T}^2$ and hence A_T is a projection.

For any $x \in \mathcal{H}_T^{(u)}$, $x = {A_T}^2 x \in A_T\mathcal{H}$ and $\mathcal{H}_T^{(u)} \subseteq A_T\mathcal{H}$.

For any $x \in \mathcal{H}$ and for each $n = 1, 2, \cdots$,

$$\|T^n A_T x\| = \|A_T T^n x\| = \|A_T x\| \quad \text{by Lemma 1}$$

and

$$\begin{aligned}\|T^{*n} A_T x\| &= \|T^* A_T T^{*n-1} x\| = \|A_T T^{*n-1} x\| \\ &= \|T^* A_T T^{*n-2} x\| = \|A_T T^{*n-2} x\| \\ &= \cdots = \|A_T x\|\end{aligned}$$

and hence $A_T\mathcal{H} \subseteq \mathcal{H}_T^{(u)}$. Therefore $\mathcal{H}_T^{(u)} = A_T\mathcal{H}$.

Conversely if A_T is the projection from $\mathcal{H}$ onto $\mathcal{H}_T^{(u)}$, then $A_T\mathcal{H}$ reduces T and $T|_{A_T\mathcal{H}}$ is unitary and hence $\|T^* A_T x\| = \|A_T x\|$ for all $x \in \mathcal{H}$. Therefore $A_T T$ is normal by Lemma 1. □

3. Conclusion. Now we can generalize Proposition as follows.

Theorem. *If a contraction T on $\mathcal{H}$ belongs to the class $\mathcal{F}$, then A_{T^*} is the projection from $\mathcal{H}$ onto $\mathcal{H}_T^{(u)}$.*

Proof. By Lemma 3, we have only to prove that $A_{T^*}T^*$ is normal.

Let $A_{T^*}T^* = V_{T^*}A_{T^*}$ is the polar decomposition of $A_{T^*}T^*$. Then, by Lemma 2, $[A_{T^*}\mathcal{H}]^\sim$ reduces V_{T^*} and

$$W = V_{T^*}|_{[A_{T^*}\mathcal{H}]^\sim} \oplus I_{[A_{T^*}\mathcal{H}]^\perp} \quad \text{on} \quad \mathcal{H} = [A_{T^*}\mathcal{H}]^\sim \oplus [A_{T^*}\mathcal{H}]^\perp$$

is an isometry on $\mathcal{H}$, where $I_{[A_{T^*}\mathcal{H}]^\perp}$ denotes the identity operator on $[A_{T^*}\mathcal{H}]^\perp$, and

(1) $$A_{T^*}T^* = V_{T^*}A_{T^*} = WA_{T^*}.$$

Since, by (1),

(2) $$A_{T^*}W^* = TA_{T^*},$$

we have, by the assumption that $T \in$ the class $\mathcal{F}$,

(3) $$A_{T^*}W = T^* A_{T^*}.$$

And since

$$(A_{T^*}W^*)W^* = (TA_{T^*})W^* = T(A_{T^*}W^*) \quad \text{by (2)},$$

we have, by the same reason as above,

$$A_{T^*} = (A_{T^*}W^*)W = T^*(A_{T^*}W^*) = (A_{T^*}W)W^* \quad \text{by (3)}$$

and

$$\left[I_{[A_{T^*}\mathcal{H}]^{\sim}} - (V_{T^*}|_{[A_{T^*}\mathcal{H}]^{\sim}})(V_{T^*}|_{[A_{T^*}\mathcal{H}]^{\sim}})^*\right] A_{T^*}\mathcal{H} = (I_{\mathcal{H}} - WW^*)A_{T^*}\mathcal{H} = \{o\}$$

and hence $V_{T^*}|_{[A_{T^*}\mathcal{H}]^{\sim}}$ is unitary.

Since

$$A_{T^*}T^* = V_{T^*}A_{T^*} \quad \text{by (1)}$$

and since

$$A_{T^*}T = W^*A_{T^*} = V_{T^*}{}^*A_{T^*} \quad \text{by (3),}$$

we have

$$A_{T^*}{}^2V_{T^*} = A_{T^*}(T^*A_{T^*}) = (A_{T^*}T^*)A_{T^*} = V_{T^*}A_{T^*}{}^2$$

and V_{T^*} commutes with A_{T^*} and hence $A_{T^*}T^* = V_{T^*}A_{T^*}$ is normal. □

Corollary 1. *If T is a contraction such that, for some positive integer n, T^n belongs to the class $\mathcal{F}$, then A_{T^*} is the projection from $\mathcal{H}$ onto $\mathcal{H}_T^{(u)}$.*

Proof. Since $A_{T^{*n}} = A_{T^*}$ and since $A_{T^n} = A_T$, $\mathcal{H}_{T^n}^{(u)} = \mathcal{H}_T^{(u)}$ and hence the conclusion follows from Theorem. □

Corollary 2. *If T is a contraction such that, for some positive integer n, T^n belongs to the class $\mathcal{F}$, then $A_T = I_{\mathcal{H}^{(u)}} \oplus B$ for some positive contraction B on $\mathcal{H} \ominus \mathcal{H}^{(u)}$.*

Proof. For any $x \in \mathcal{H}$, let $x = A_{T^*}x + (I - A_{T^*})x$. Then, by Theorem, A_{T^*} is the projection from $\mathcal{H}$ onto $\mathcal{H}_T^{(u)}$ and it commutes with T and hence, for any positive integer m, we have

$$\begin{aligned}\|T^m x\|^2 &= \|T^m A_{T^*}x\|^2 + \|T^m(I - A_{T^*})x\|^2 \\ &\geq \|T^m A_{T^*}x\|^2 = \|A_{T^*}x\|^2.\end{aligned}$$

Therefore we have $A_T{}^2 \geq A_{T^*}{}^2$ and $A_T \geq A_{T^*}$ by Heinz's inequality. Since A_{T^*} commutes with T, A_{T^*} commutes with A_T and $A_T = I_{\mathcal{H}^{(u)}} \oplus B$ for some positive contraction B on $\mathcal{H} \ominus \mathcal{H}^{(u)}$ because $A_T \leq I$. □

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