On the mean value of $|L(1,\chi)|^2$ for odd primitive Dirichlet characters

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Abstract: Let $f > 1$ be given. Whereas a simple formula for the mean value of $|L(1,\chi)|^2$ for odd Dirichlet characters modulo f is known, we explain why there is no hope of ever finding a simple formula for the mean value of $|L(1,\chi)|^2$ for primitive odd Dirichlet characters modulo f.

Key words: Dirichlet characters; L-functions.

1. Introduction. Let $f > 1$ be given. Let X_f^- and P_f^- denote the set of all the odd Dirichlet characters modulo f and of all the primitive odd Dirichlet characters modulo f, respectively (see [1, Section 6.8] for the definition of Dirichlet characters, see [1, Section 8.7] for the definition of primitivity and recall that an odd Dirichlet character is a Dirichlet character χ which satisfies $\chi(-1) = -1$). Whenever $d > 0$ divides f we let $\tilde{\psi} \in X_f^-$ denote the character induced by $\psi \in X_{f/d}^-$. Since $\chi \in X_f^-$ is not primitive if and only if there exist a prime p dividing f and $\psi \in X_{f/d}^-$ such that $\chi = \tilde{\psi}$ is induced by ψ, for any complex s we get (use the inclusion-exclusion principle) :

$$(1)\quad \sum_{\chi\in P_f^-} |L(s,\chi)|^2 = \sum_{d|f} \mu(d) \sum_{\psi\in X_{f/d}^-} |L(s,\tilde{\psi})|^2$$

where μ and ϕ denote the Möbius and Euler totient functions (see [1, Chapter 2]) and $L(s,\chi)$ denotes the Dirichlet L-functions associated with χ (see [1, Chapter 11]). Notice that $\#X_1^- = \#X_2^- = 0$ and $\#X_f^- = \phi(f)/2$ whenever $f > 2$. We proved:

Theorem 1 (See [2], [3]). *It holds*

$$(2)\quad \sum_{\chi\in X_f^-} |L(1,\chi)|^2 = \frac{\pi^2\phi(f)}{12}\prod_{p|f}\left(1-\frac{1}{p^2}\right) - \frac{\pi^2\phi^2(f)}{4f^2}.$$

We deduce:

Corollary 2 (See [5]). *If $f > 1$ is square-full then it holds*

$$(3)\quad \sum_{\chi\in P_f^-} |L(1,\chi)|^2 = \frac{\pi^2\phi^2(f)}{12f}\prod_{p|f}\left(1-\frac{1}{p^2}\right).$$

Proof. If $d > 0$ is square-free and divides f and $\psi \in X_{f/d}^-$ then $L(s,\tilde{\psi}) = L(s,\psi)$ (use the Euler products of both these terms (see [1, Section 11.5])). Hence, (1) yields

$$\sum_{\chi\in P_f^-} |L(1,\chi)|^2 = \sum_{d|f} \mu(d) \sum_{\psi\in X_{f/d}^-} |L(1,\psi)|^2,$$

and the desired result follows from Theorem 1. □

It was conjectured (not in contradiction with (3)) that:

Conjecture 3 (See [MR 91j:11068] and [5]). *For any rational integer $f > 1$ we have*:

$$(4)\quad \sum_{\chi\in P_f^-} |L(1,\chi)|^2 = \frac{\pi^2}{12}\frac{\phi(f)}{f}\frac{J(f)}{f}\left(f\prod_{p|f}\left(1+\frac{1}{p}\right) + 2\mu(f)\right)$$

where $J(f) = \sum_{d|f}\mu(d)\phi(f/d)$ is the number of primitive characters modulo f.

This conjecture is false. Indeed, if $f = 15$ then P_{15}^- is reduced to the character $n \mapsto \chi(n) = (n/15)$ (Jacobi's symbol) for which $L(1,\chi) = 2\pi/\sqrt{15}$ (for the class number of the imaginary quadratic field $\mathbf{Q}(\sqrt{-15})$ is equal to 2), the left hand side of (4) is equal to $4\pi^2/15$ while the right hand side of (4) is equal to $52\pi^2/15^2$. Not only is this conjecture false, but its falsity does not trivially comes from any misprint in (4) for according to such a conjecture, $S^-(pq)$ defined below should be polynomial in p and q whenever p and q range over the positive rational primes, whereas we will prove:

Theorem 4. *Let p and q denote distinct positive primes. Even though*

$$S^-(pq) \stackrel{\text{def}}{=} \frac{(pq)^3}{\pi^2}\sum_{\chi\in P_{pq}^-}|L(1,\chi)|^2$$

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is always a positive rational number, there does not exist any polynomial $f(X,Y)$ such that for all pairs (p,q) we have $S^-(pq) = f(p,q)$.

Therefore, it seems that there is no hope of ever finding a neat explicit formula for the sums $\sum_{\chi\in P_f^-}|L(1,\chi)|^2$ which would be valid for any $f>1$.

2. Proof of Theorem 4.

Theorem 5. *Whenever $d\geq 1$ divides $f>1$ we set*

$$(5)\qquad T_\pm(f,d)\stackrel{\text{def}}{=}\sideset{}{^*}\sum_{1\leq a\leq f}\ \sideset{}{^*}\sum_{\substack{1\leq b\leq f\\ b\equiv\pm a\pmod{f/d}}} ab,$$

(where $\sum^*$ stands for a summation ranging over indices relatively prime to f). We have

$$(6)\qquad S^-(f)\stackrel{\text{def}}{=}\frac{f^3}{\pi^2}\sum_{\chi\in P_f^-}|L(1,\chi)|^2 = \sum_{d|f}\mu(d)\phi(f/d)T_+(f,d),$$

and $S^-(f)$ is always a positive rational integer. Notice that if p and q denote distinct positive primes then (6) yields

$$(7)\qquad S^-(pq)=\phi(pq)T_+(pq,1)-\phi(q)T_+(pq,p) -\phi(p)T_+(pq,q)+T_+(pq,pq).$$

Proof.

$$S^-(f)= f^2\sum_{\chi\in P_f^-}|L(0,\chi)|^2$$

(use the functional equation satisfied by $L(s,\chi)$ (see [1, Section 12.10]))

$$= f^2\sum_{d|f}\mu(d)\sum_{\psi\in X_{f/d}^-}|L(0,\tilde\psi)|^2$$

(use (1) for $s=0$)

$$=\sum_{\substack{d|f\\ d<f/2}}\mu(d)\sum_{\psi\in X_{f/d}^-}\left|\sum_{a=1}^{f}a\tilde\psi(a)\right|^2$$

(use [1, Section 12.13])

$$=\frac12\sum_{\substack{d|f\\ d<f/2}}\mu(d)\phi(f/d)(T_+(f,d)-T_-(f,d))$$

$$=\frac12\sum_{d|f}\mu(d)\phi(f/d)(T_+(f,d)-T_-(f,d))$$

(for $T_+(f,f)=T_-(f,f)$ and $T_+(f,f/2)=T_-(f,f/2)$ whenever f is even)

where we have used

$$\sum_{\psi\in X_{f/d}^-}\tilde\psi(a)\overline{\tilde\psi(b)}=\sum_{\psi\in X_{f/d}^-}\psi(a)\overline{\psi(b)} =\begin{cases}\phi(f/d)/2 & \text{if } b\equiv a\pmod{f/d}\\ -\phi(f/d)/2 & \text{if } b\equiv -a\pmod{f/d}\\ 0 & \text{otherwise}\end{cases}$$

(provided that $\mathrm{pgcd}(a,f)=\mathrm{pgcd}(b,f)=1$). Now, since the canonical morphism $s:(\mathbf{Z}/f\mathbf{Z})^*\longrightarrow(\mathbf{Z}/(f/d)\mathbf{Z})^*$ is surjective, for any given a relatively prime to f we have

$$\sideset{}{^*}\sum_{1\leq a\leq f}\ \sideset{}{^*}\sum_{\substack{1\leq b\leq f\\ b\equiv a\pmod{f/d}}} a=\#\ker s\cdot\sideset{}{^*}\sum_{1\leq a\leq f}a =\frac{\phi(f)}{\phi(f/d)}\sideset{}{^*}\sum_{1\leq a\leq f}a=\frac{f\phi^2(f)}{2\phi(f/d)}$$

and

$$T_-(f,d)=\sideset{}{^*}\sum_{1\leq a\leq f}\ \sideset{}{^*}\sum_{\substack{1\leq b\leq f\\ b\equiv a\pmod{f/d}}} a(f-b) =\frac{f^2\phi^2(f)}{2\phi(f/d)}-T_+(f,d),$$

which provides us with the desired result in using $\sum_{d|f}\mu(d)=0$. □

Lemma 6. *Whenever $q=np+1$ and p are prime, it holds $S^-(p,q)=g(p,q)$ where $g(X,Y)\stackrel{\text{def}}{=}X^2Y^2(X-1)^2(Y-1)^2/12-XY^2(X-1)(Y-1)^2/6$.*

Proof. Using

$$T_+(f,1)=\sideset{}{^*}\sum_{1\leq a\leq f}a^2=\frac13 f^2\phi(f)+\frac16 f\prod_{p|f}(1-p)$$

we obtain $T_+(pq,1)=p^2q^2(p-1)(q-1)/3+pq(p-1)(q-1)/6$, and using

$$T_+(f,f)=\left(\sideset{}{^*}\sum_{1\leq a\leq f}a\right)^2=\frac14 f^2\phi^2(f)$$

we obtain $T_+(pq,pq)=p^2q^2(p-1)^2(q-1)^2/4$. Now, writting $a=A+qA'$ and $b=A+qB'$ with $1\leq A\leq q$, $0\leq A'\leq p-1$ and $0\leq B'\leq p-1$, we get

$$T_+(pq,p)=\sum_{A=1}^{q-1}\left(\sum_{\substack{A'=0\\ \mathrm{pgcd}(A+qA',p)=1}}^{p-1}A+qA'\right)^2.$$

Then, we notice that p divises $A+qA'$ if and only if $A' \equiv -A \pmod{p}$, and we write $A = pQ+R$ with $1 \le R \le p$ and $Q \ge 0$. We get

$$\begin{aligned} &T_+(pq,p) \\ &= \sum_{Q=0}^{n-1}\sum_{R=1}^{p}\left(\sum_{\substack{A'=0 \\ A'\not\equiv -R \pmod{p}}}^{p-1} pQ+R+qA'\right)^2 \\ &= \sum_{Q=0}^{n-1}\sum_{R=1}^{p}\left(\sum_{\substack{A'=0 \\ A'\neq p-R}}^{p-1} pQ+R+qA'\right)^2 \\ &= \sum_{Q=0}^{n-1}\sum_{R=1}^{p}\left(p(p-1)Q+(p+q-1)R+\frac{p-3}{2}pq\right)^2 \\ &= p^2q^2(p-1)^2(q-1)/4 \\ &\quad + pq(p-1)(q-1)(p+q-1)/6. \end{aligned}$$

In the same way,

$$T_+(pq,q) = \sum_{A=1}^{p-1}\left(\sum_{\substack{A'=0 \\ \mathrm{pgcd}(A+pA',q)=1}}^{q-1} A+pA'\right)^2$$

and q divides $A+pA'$ if and only if q divides $nA + npA'$, hence if and only if $A' \equiv nA \pmod{q}$. Since $0 \le nA \le n(p-1) < q$, then q divides $A+pA'$ if and only if $A' = nA$, which yields $A+pA' = A+pnA = qA$. Hence,

$$\begin{aligned} T_+(pq,q) &= \sum_{A=1}^{p-1}\left(-qA+\sum_{A'=0}^{q-1}(A+pA')\right)^2 \\ &= \sum_{A=1}^{p-1}\left(pq\frac{q-1}{2}\right)^2 = p^2q^2(p-1)(q-1)^2/4. \end{aligned}$$

The Lemma follows from (7) and these four previous formulae. □

Now, we are in a position to prove the last assertion of Theorem 4: Lemma 6 would give $f(X,Y) = g(X,Y)$ (according to Dirichlet's Theorem, for any prime p there are infinitely many primes q of the form $q = np+1$, $n \ge 1$. Hence, for any prime p we would have $f(p,Y) = g(p,Y)$. Now, since there are infinitely many primes $p > 2$ we would then obtain $f(X,Y) = g(X,Y)$). But this identity cannot hold for while $S^-(pq) = S^-(qp)$, this expression $g(p,q)$ is not symetrical in p and q.

References

[1] T. M. Apostol: Introduction to Analytic Number Theory. Undergrad. Texts Math., Springer-Verlag, pp. 1–338 (1976).

[2] S. Louboutin: Quelques formules exactes pour des moyennes de fonctions L de Dirichlet. Canad. Math. Bull., **36**, 190–196 (1993).

[3] S. Louboutin: Corrections à: Quelques formules exactes pour des moyennes de fonctions L de Dirichlet. Canad. Math. Bull., **37**, 89 (1994).

[4] Qi Minggao: A kind of mean square formula for L-functions. J. Tsinghua Univ., **31**, 34–41 (1991) (see MR 93g:11090).

[5] W. P. Zhang: A note on a class of mean square values of L-functions. J. Northwest Univ., (3) **20**, 9–12 (1990) (see MR 91j:11068).