Quasilinear degenerate elliptic equations with absorption term

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1. Introduction. Let $N \geq 1$ and $p > 1$. Let F be a given **smooth compact set** and Ω be a bounded open set of $\mathbf{R}^N$ satisfying $F \subset \Omega \subset \mathbf{R}^N$ and $F \neq \phi$. We also set $\Omega' = \Omega \setminus \partial F$, where $\partial F = F \setminus \overset{\circ}{F}$. Here by $\overset{\circ}{F}$ we denote the interior of F, which may be empty.

By $H^{1,p}(\Omega)$ we denote the space of all functions on Ω, whose generalized derivatives $\partial^\gamma u$ of order ≤ 1 satisfy

$$(1\text{-}1) \quad ||u||_{1,p} = \sum_{|\gamma| \leq 1} \left(\int_\Omega |\partial^\gamma u(x)|^p \, dx \right)^{1/p} < +\infty,$$

and by $H^{1,p}_{\mathrm{loc}}(\Omega)$ the local version of $H^{1,p}(\Omega)$. For $u \in H^{1,p}_{\mathrm{loc}}(\Omega')$, we define a generalized p-harmonic operator by

$$(1\text{-}2) \qquad L_p u = -\operatorname{div}(A(x)|\nabla u|^{p-2}\nabla u),$$

where $\nabla u = (\partial u/\partial x_1, \partial u/\partial x_2, \ldots, \partial u/\partial x_N)$, and $A(x) \in C^1(\Omega')$ is positive in $\Omega \setminus F$ and vanishes in $\overset{\circ}{F}$. We shall study the Dirichlet boundary problem for the genuinely degenerate elliptic operators L_p with absorption term:

$$(1\text{-}3) \quad \begin{cases} L_p u + B(x)Q(u) = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Here $B(x)$ is a nonnegative function on Ω, and $Q(t)$ is a continuous and strictly monotone increasing function on $\mathbf{R}$. In connection with this problem we shall treat two topics in the present paper. Namely, one is concerned with removable singularities of solutions for (1-3) and the other is the unique existence property of bounded solutions. We note that if $p = 2$, then these topics were already treated in the author's paper [6] under a similar framework. By virtue of Kato's inequality and a maximum principle, the unique existence of bounded solutions was established. Since Kato's inequality does not work effectively in the quasilinear case, we shall employ in this paper a priori estimates, a comparison principle and a weak maximum principle instead. Since the operators L_p are rather general, we need to modify them suitably so that they are applicable to our problems.

2. Preliminaries. In this section we prepare our basic framework and some notations which are of importance through the present paper.

Let $N \geq 1$ and $p > 1$. Let F and Ω be a nonempty smooth compact set and bounded open suset of $\mathbf{R}^N$ respectively, satisfying $F \subset \Omega$, and set $\Omega' = \Omega \setminus \partial F$. Here ∂F is defined as $\partial F = F \setminus \overset{\circ}{F}$. In the next we define a distance to ∂F.

Definition 1. By $d(x)$ we denote a distance function $d(x) = \operatorname{dist}(x, \partial F)$.

Remark. A distance function $d(x)$ is Lipschitz continuous and differentiable almost everywhere. Moreover one can approximate it by a smooth function. Namely there exists a nonnegative smooth function $D(x) \in C^\infty(\Omega')$ such that

$$(2\text{-}1) \quad \begin{aligned} &C(0)^{-1} \leq \frac{D(x)}{d(x)} \leq C(0), \\ &|\partial^\gamma D(x)| \leq C(|\gamma|) d(x)^{1-|\gamma|}, \quad x \in \Omega', \end{aligned}$$

where γ is an arbitrary multi-index and $C(|\gamma|)$ is a positive number depending on $|\gamma|$. Therefore one can assume that $d(x)$ is smooth as well without loss of generality. (For the construction of $D(x)$, see [9] for example.)

First we assume the following **(H-1)** on nonnegative functions $A(x)$ and $B(x)$.

(H-1)

$$(2\text{-}2) \quad \begin{cases} A(x) \in C^1(\Omega') \cap L^1_{\mathrm{loc}}(\Omega), \\ A(x) = 0 \quad \text{in} \quad \overset{\circ}{F} = F \setminus \partial F, \\ A(x) > 0 \quad \text{in} \quad \Omega \setminus F, \\ B(x) \in L^\infty_{\mathrm{loc}}(\Omega') \cap L^1_{\mathrm{loc}}(\Omega), \\ B(x) > 0 \quad \text{in} \quad \Omega' = \Omega \setminus \partial F. \end{cases}$$

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Secondly we assume the following **(H-2)** on the nonlinear term $Q(t)$.

(H-2)

$Q(t)$ is a monotone increasing and continuous function such that $Q(0) = 0$ and $t \cdot Q(t) > 0$ on $\mathbf{R} \setminus \{0\}$. Moreover we assume that there is a positive number δ_0 such that

$$\limsup_{|t|\to\infty} \frac{|t|^{p-1+\delta_0}}{|Q(t)|} < +\infty. \tag{2-3}$$

We need more notations.

Definition 2. Let δ_0 be a positive number. Let us set for any $t > 0$ and any $x \in \Omega' = \Omega \setminus \partial F$,

$$\begin{cases} \tilde{A}(x) = A(x) + d(x)|\nabla A(x)|, \\ \mathbf{M}(x) = \text{ess-sup}_{\{y\in\Omega:1/4<d(y)/d(x)<3\}} \dfrac{\tilde{A}(y)}{B(y)}, \\ K(x,t) = 1 + (\mathbf{M}(x) \cdot t^p)^{p-1/\delta_0}. \end{cases} \tag{2-4}$$

The following assumption is crucial in the present work.

[H-3]

For the same positive number $\delta_0>0$ as in **[H-2]**, it holds that

(2-5)

$$\liminf_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^p} \int_{\varepsilon/2<d(x)<\varepsilon} A(x)K\left(x, \frac{1}{d(x)}\right) dx < +\infty.$$

Remark. Originally the definition of the kernel $K(x,t)$ comes from the pointwise estimate of the supersolutions of the equation (1-3) under some additional assumptions (see Lemma 7-1). More precisely, we can show that every solution u of (1-3) in $H^{1,p}_{\text{loc}}(\Omega')\cap L^\infty_{\text{loc}}(\Omega')$ is dominated by $K(x,1/d)^{1/(p-1)}$ up to multiplication by constants. Roughly speaking, the condition **(H-3)** guarantees the integrability of the term $B{\cdot}Q(u)$ near ∂F with u being the solution of (1-3), and then we can finally show the boundedness of the solution, which is one of the main purpose in the present paper. It is very interesting that we can reconstruct the kernel without making use of the explicit supersolutions. Roughly speaking, the kernel $K(x,t)$ is equivalent to the conjugate function of the nonlinear term Q, if it is strictly convex. For the further information see [8].

3. Removable singularities. Let D be an open subset of $\mathbf{R}^N$. By $\mathcal{D}'(D)$ we denote the space of all distributions on D.

Theorem 1 (Removable singularities). *Assume* **(H-1)**, **(H-2)** *and* **(H-3)**. *Assume that $C(x) \in L^\infty_{\text{loc}}(\Omega') \cap L^1_{\text{loc}}(\Omega)$ satisfies for some positive number C*

$$0 \le C(x) \le C \cdot B(x), \quad \textit{for almost all } x \in \Omega. \tag{3-1}$$

Assume that $u \in H^{1,p}_{\text{loc}}(\Omega') \cap L^\infty_{\text{loc}}(\Omega')$ satisfies $L_p u \in L^1_{\text{loc}}(\Omega')$ in the distribution sense. Moreover we assume that for almost all $x \in \{x \in \Omega'; u(x) \ge 0\}$,

$$L_p u + B(x)Q(u) \le C(x). \tag{3-2}$$

Then we have $u_+ \in L^\infty_{\text{loc}}(\Omega)$, where $u_+ = \max(u,0)$.

As a corollary, we have

Theorem 2. *Assume* **(H-1)**, **(H-2)** *and* **(H-3)**. *Assume that $f(x) \in L^\infty_{\text{loc}}(\Omega') \cap L^1_{\text{loc}}(\Omega)$ satisfies for some positive number C*

$$|f(x)| \le C \cdot B(x), \quad \textit{for almost all } x \in \Omega. \tag{3-3}$$

Assume that $u \in H^{1,p}_{\text{loc}}(\Omega') \cap L^\infty_{\text{loc}}(\Omega')$ satisfies

$$L_p u + B(x)Q(u) = f, \quad \textit{in } \mathcal{D}'(\Omega'). \tag{3-4}$$

Then there exists a function $v \in L^\infty_{\text{loc}}(\Omega)$ such that

$$\begin{cases} L_p v + B(x)Q(v) = f, \quad \textit{in } \mathcal{D}'(\Omega) \\ v\big|_{\Omega'} = u. \end{cases} \tag{3-5}$$

Proof. This is a direct consequence of Theorem 1. In fact from Theorem 1 we have $u_+ \in L^\infty_{\text{loc}}(\Omega)$. And similarly $u_- \in L^\infty_{\text{loc}}(\Omega)$. Thus $u \in L^\infty_{\text{loc}}(\Omega)$. Here we note that since $A(x) = 0$ on $F \setminus \partial F$, $u(x) = v(x) = Q^{-1}(f(x)/B(x)))$ on $F \setminus \partial F$. Then u can be extended as a solution of the same equation in whole Ω. Here we note that the uniqueness of solutions in $L^\infty_{\text{loc}}(\Omega)$ follows from the same argument in the proof of Theorem 3. □

4. Existence and uniqueness of solutions for Dirichlet problem.

Theorem 3 (Dirichlet problem). *Assume* **(H-1)**, **(H-2)** *and* **(H-3)**. *Assume that $f(x) \in L^\infty(\Omega)$ satisfies for some positive number C*

$$|f(x)| \le C \cdot B(x), \quad \textit{for almost all } x \in \Omega. \tag{4-1}$$

Moreover we assume that $A(x), B(x) \in C^0(\overline{\Omega})$. Then there exists a unique function

$$u \in L^\infty(\Omega) \cap H^{1,p}_{\text{loc}}(\overline{\Omega} \setminus F) \tag{4-2}$$

which satisfies (1-3) *in the distribution sense and satisfies*

$$\int_\Omega [A(x)|\nabla u|^p + B(x)Q(u)u]\, dx \le C'[||f/B||^\lambda_\infty + ||f||_\infty]. \tag{4-3}$$

Here $\lambda = (p+\delta_0)/(p-1+\delta_0)$ and C' is a positive number independent of each function f.

Remark. If Q is uniformly Lipschitz continuous, then $u \in H^{2,p}_{\mathrm{loc}}(\Omega \setminus F)$ as well. For the proof of this Theorem 3 we shall regularize the problem. By virtue of Theorems 1 and 2, we shall prove that the unique solutions of these approximating nonlinear elliptic equations converge to the unique bounded solution of the original equation. Here we note that the operator L_p itself is not ε-regularizable, because it may be degenerate infinitely on ∂F.

Remark. If we assume that ∂F is smooth, then we can also establish the Hölder continuity of the gradient$|\nabla u|$ of the solution u under some additional contitions. More precisely, in the coming paper we shall show $|\nabla u|$ belongs to the weighted Schauder space if $A(x)$ is of class Muckenhoupt's A_p class and $A(x)$ is a power of the distance to ∂F.

5. Example. In this section we construct examples to illustrate Theorem 1. Let F be either the origin 0 or an $m-$dimensional C^∞ compact submanifolds in $\mathbf{R}^N$ with $0 < m \leq N-1$, and let $d(x)$ be a distance function defined by (2-1). If F consists of the origin 0, then we put $d(x) = |x|$. For $p > 1$ and $q > p-1$, we set

$$\text{(5-1)} \qquad Pu = -\operatorname{div}(d(x)^{p\alpha}|\nabla u|^{p-2}\nabla u) + b(x)\cdot d(x)^{p\beta}\cdot |u|^{q-1}u,$$

where $b(x)$ is a positive continuous function. Then the condition **(H-1)** is equivalent to the following **(h-1)**.

$$\textbf{(h-1)} \qquad \min(\alpha,\beta) > -\frac{N-m}{p}.$$

Let us set for $0 \leq m \leq N-1$

$$\text{(5-2)} \quad p^*_m = \begin{cases} (p-1)\cdot\left(1 + p\dfrac{1-\alpha+\beta}{N+p\alpha-p-m}\right), \\ \qquad\qquad \text{if} \quad \alpha < \beta+1, \\ p-1, \qquad \text{if} \quad \alpha \geq \beta + 1. \end{cases}$$

Here we note that **(H-2)** is satisfied automatically for $\delta_0 = q-p+1 > 0$. Then **(H-3)** is equivalent to **(h-2)**.

$$\textbf{(h-2)} \quad \begin{cases} q \geq p^*_m, & \text{if} \quad \alpha < \beta+1, \\ q > p^*_m = p-1, & \text{if} \quad \alpha \geq \beta+1, \\ \alpha > -\dfrac{N-m-p}{p}. \end{cases}$$

Remark. For this operator P one can show a sharper result than Theorem 1. In fact, it is not difficult to see that the condition **(h-2)** is necessary for the validity of Theorem 1 under the assumptions **(h-1)** and $q > p-1$. For the precise result, see [7] and [8].

6. Auxiliary lemmas. In this section we shall describe auxiliary lemmas concerning basic estimates for weak solutions of the equation, which will be needed to establish Theorems stated in §3 and §4. Without loss of generality we assume that $\{x : d(x) < 3\} \subset \Omega$.

Lemma 6-1 (A priori inequality 1). *Assume* **(H-1)** *and* **(H-2)**. *Assume that $f(x) \in L^\infty_{\mathrm{loc}}(\Omega') \cap L^1_{\mathrm{loc}}(\Omega)$ satisfies for some positive number C*

$$\text{(6-1)} \quad |f(x)| \leq C\cdot B(x), \quad \textit{for almost all } x \in \Omega.$$

Assume that $u \in H^{1,p}_{\mathrm{loc}}(\Omega') \cap L^\infty(\Omega')$ satisfies

$$\text{(6-2)} \quad L_p u + B(x)\cdot Q(u) = f, \quad \textit{in } D'(\Omega').$$

Then we have, for any number $q > 0$ and any function $\eta \in C^\infty_0(\Omega \setminus F)$,

$$\text{(6-3)} \int_\Omega A(x)|\nabla(u-\mu)_+|^p (u-\mu)_+^{q-1}\eta^p\,dx \leq \leq \left(\frac{p}{q}\right)^p \int_\Omega A(x)|\nabla\eta|^p (u-\mu)_+^{p+q-1}\,dx$$

Here μ is an arbitrary positive number satisfying

$$\text{(6-4)} \quad Q(\mu) \geq \max\left[\sup_{x\in\Omega}\frac{f(x)}{B(x)}, \sup_{2/3<d(x)<1}|u|\right].$$

Lemma 6-2 (A priori inequality 2). *Assume the same assumptions as in* Lemma 6-1.

Then we have, for any number $q > 0$ and any function $\eta \in C^\infty_0(\Omega \setminus F)$,

$$\text{(6-5)} \quad \int_\Omega [B(x)Q(u) - f(x)](u-\mu)_+^q\,\eta^p\,dx \leq \frac{p^p}{q^{p-1}}\int_\Omega A(x)|\nabla\eta|^p(u-\mu)_+^{p+q-1}\,dx,$$

$$\text{(6-6)} \quad \int_{\{x;u\geq\mu\}} [B(x)Q(u) - f(x)]\,\eta^p\,dx \leq p^p\left(\int_\Omega A(x)|\nabla\eta|^p(u-\mu)_+^p\,dx\right)^{(p-1)/p} \times\left(\int_\Omega A(x)|\nabla\eta|^p\,dx\right)^{1/p}.$$

Here μ is an arbitrary positive number satisfying (6-4).

7. A sketch of the proof of Theorem 1. First we show an a priori bound for weak solutions of (1.3).

Lemma 7-1 (Supersolution). *Assume that* $u \in H^{1,p}_{\mathrm{loc}}(\Omega') \cap L^{\infty}_{\mathrm{loc}}(\Omega')$ *satisfies* $L_p u \in L^1_{\mathrm{loc}}(\Omega')$ *in the distribution sense. Assume that* **(H-1)** *and* **(H-2)**, *and assume that* $C(x) \in L^{\infty}_{\mathrm{loc}}(\Omega') \cap L^1_{\mathrm{loc}}(\Omega)$ *satisfies for some positive number* C

(7-1) $0 \le C(x) \le C \cdot B(x)$, *for almost all* $x \in \Omega$.

Moreover we assume that for almost all $x \in \{x \in \Omega;\ u(x) \ge 0\}$

(7-2) $$L_p u + B(x)Q(u) \le C(x).$$

Then we have, for some positive numbers C_1 *and* ε_0,

(7-3) $$u(x) \le C_1 \cdot K\left(1, \frac{1}{d(x)}\right)^{1/(p-1)},$$

for any x *with* $0 < d(x) \le \varepsilon_0$.

In this stage, the weak solution u of the inequality (3-1) may still have singularities on ∂F. Combining this weak result with Lemma 6-2 and the condition **(H-3)**, we are able to show that u is bounded in Ω. For the detailed proof see [8].

8. A sketch of the proof of Theorem 3.

Uniqueness. Since the operator L_p and the nonlinear term Q are monotone, the uniqueness of solutions is easy to see in the space

(8-1)
$$T(\Omega) = \{u \in L^{\infty}(\Omega) \cap H^{1,p}_{\mathrm{loc}}(\overline{\Omega} \setminus F); u = 0 \text{ on } \partial\Omega\}.$$

Existence. First we shall regularize the problem by approximating the operator L_p by uniformly elliptic operators $\{L^{(\varepsilon)}_p\}_{\varepsilon>0}$ in the following way. Let us set for $\varepsilon > 0$

(8-2) $$L^{(\varepsilon)}_p u = -\operatorname{div}[(\varepsilon + A(x))|\nabla u|^{p-2}\nabla u],$$

for $u \in H^{1,p}_0(\Omega)$, and consider the Dirichlet problem:

(8-3) $$\begin{cases} L^{(\varepsilon)}_p u + B(x)Q(u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Then we can show the existence and regularity of solutions of (8-3). By u_ε we denote the solutions to (8-3). We see $u_\varepsilon \in H^{1,p}_0(\Omega)$ and $BQ(u_\varepsilon)u_\varepsilon \in H^{1,p}_0(\Omega)$. It follows from Young's inequality and **(H-2)** that u_ε satisfies (4-3) uniformly in $\varepsilon > 0$. By the method of a priori estimate and compactness, we can derive a subsequence $\{u_{\varepsilon_j}\}_{j=1}^{\infty}$ from $\{u_\varepsilon\}_{\varepsilon>0}$ which converges weakly to some element $\overline{u} \in H^{1,p}_{\mathrm{loc}}(\overline{\Omega} \setminus F)$ and $\{u_{\varepsilon_j}\}$ converges to $\overline{u}$ a.e. in $\Omega \setminus F$. We also see that $\{Bu_{\varepsilon_j}Q(u_{\varepsilon_j})\}$ converges to $B\overline{u}Q(\overline{u})$ in $L^1_{\mathrm{loc}}(\overline{\Omega} \setminus F)$ by Fatou's lemma and a weakly lower semicontinuity of L^p-norm. Then $\overline{u}$ satisfies (1-3) in $\Omega \setminus F$ in the sense of distribution. Now we define

(8-4) $$u(x) = \begin{cases} \overline{u}(x), & \text{if } x \in \Omega \setminus F, \\ Q^{-1}(f(x)/B(x)), & \text{if } x \in F \setminus \partial F. \end{cases}$$

Then u clearly satisfies (1-3) in $\Omega \setminus \partial F$ in the sense of distribution. Hence it follows from Theorem 1 that u is bounded in Ω. From Theorem 2 we see that there exists a unique function $v \in L^{\infty}_{\mathrm{loc}}(\Omega)$ which satisfies (1-3). Since $v = u$ in $\Omega \setminus \partial F$, we see that $v \in T(\Omega)$ is the unique solution to (1-3) in $D'(\Omega)$. For the precise proof see [7] and [8].

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