Cauchy-Kovalevskaja-Nagumo type theorems for PDEs with shrinkings

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1. Introduction. By the Cauchy-Kovalevskaja theorem we know that, if $f(t,x,u,v)$ is an analytic function of $(t,x,u,v) \in \mathbf{R}^4$ then the Cauchy problem

$$(1.1) \qquad \partial_1 u(t,x) = f(t,x,u(t,x),\partial_2 u(t,x)),$$

$$(1.2) \qquad u(0,x) = 0,$$

where ∂_i is the partial differentiation in the ith variable, has a unique analytic local solution. But it seems impossible simply to replace the first order derivative $\partial_2 u(t,x)$ in the equation (1.1) by a higher order derivative $\partial_2^p u(t,x)$ with $p > 1$.

So it was quite surprising to us to learn that Augustynowicz *et al.* [2], [3] had solved the Cauchy problem for an equation of the form

$$(1.3) \quad \partial_1 u(t,x) = a(t,x)\partial_2^p u(t,\alpha(t,x)x) + g(t,x)$$

or

$$(1.4) \quad \partial_1 u(t,x) = a(t,x)\partial_2^p u(\alpha(t,x)t,x) + g(t,x).$$

In (1.3) and (1.4) it is assumed that $a(t,x)$, $g(t,x)$ and $\alpha(t,x)$ are given analytic functions of (t,x). The function α is called a *delay* and has the property that $0 < \alpha(t,x) < 1$ if $|t| + |x|$ is small. It seems that a *delay* plays the role of appeasing the disturbance caused by differentiation in x.

Our purpose here is to generalize the above mentioned results in [2], [3] as much as possible. We want to consider non-linear differential equations instead of linear ones such as (1.3) or (1.4). Also we want mainly to consider the case where the differential equation is not analytic in t.

As the first step toward these ends we consider in this note two simple PDEs of the form

$$(1.5) \quad \partial_1 u(t,x) = f(t,x,u(t,x),\partial_2^p u(t,\alpha(t,x)x))$$

and

$$(1.6) \quad \partial_1 u(t,x) = f(t,x,u(t,x),\partial_2^p u(\alpha(t)t,x)).$$

In (1.5) and (1.6) $\alpha(t,x)$ or $\alpha(t)$ is a function with properties similar to those of $\alpha(t,x)$ in (1.3) and (1.4). We call $\alpha(t,x)$ or $\alpha(t)$ a *shrinking* instead of a *delay*, since it may sound strange to use the word *delay* for the space variable. Our result for the equation (1.5) will be stated in §2 as Theorem 2.1. Our result for the equation for (1.6) will be stated in §3 as Theorem 3.2. Note that the discussion for the equation (1.5) is much simpler than that for the equation (1.6). For instance, the domain of existence of a solution to the Cauchy problem (1.6)-(1.2) is of the strange form $\{(t,x)\,;\,|t| < a,\ R - |x| - |t|^{1/(p+1)} > 0\}$, while the corresponding domain for the problem (1.5)-(1.2) is of the simple form $\{(t,x)\,;\,|t| < a,\ |x| < R\}$. In order to obtain the result of §3 we use a variant of the famous trick in Nagumo [1].

For the reason why $\alpha(t)$ in (1.6) cannot be replaced by $\alpha(t,x)$ and further possibilities of obtaining other related results see the remarks in §4.

2. Shrinking in the space variable. The purpose of this section is to solve the Cauchy problem for the equation (1.5). Our result will be stated in the Theorem 2.1 below. In the theorem we use the following notation. If T, R and S are positive constants, we write

$$A(T,R) = \{(t,x) \in \mathbf{R} \times \mathbf{C}\,;\,|t| < T,\ |x| < R\},$$
$$B(T,R,S) = \{(t,x,u,v) \in \mathbf{R} \times \mathbf{C}^3\,;\ (t,x) \in A(T,R),\ |u| < S,\ |v| < S\}.$$

Theorem 2.1. *Let T, R, S and m be positive constant. Assume that $m < 1$. Let p be an integer ≥ 1. In the differential equation* (1.5) *assume that*

(i) *$f(t,x,u,v)$ is a complex valued bounded continuous function of $(t,x,u,v) \in B(T,R,S)$;*
(ii) *$f(t,x,u,v)$ is analytic in (x,u,v);*
(iii) *$\alpha(t,x)$ is a complex valued continuous function of $(t,x) \in A(T,R)$;*
(iv) *$\alpha(t,x)$ is analytic in x and satisfies the inequality*

$$|\alpha(t,x)| \leq m.$$

Then there is a positive constant a such that the Cauchy problem (1.5)-(1.2) *has a unique solution*

$u : A(a,R) \to \mathbf{C}$ such that $u(t,x)$, $\partial_1 u(t,x)$ and $\partial_2^p u(t,x)$ are continuous in (t,x) and analytic in x.

Proof . Let $a \le T$ be a positive constant and assume that there is a solution $u : A(a,R) \to \mathbf{C}$ of the Cauchy problem (1.5)-(1.2). Then

$$w(t,x) := \partial_1 u(t,x)$$

satisfies the integral equation

$$w(t,x) = f\left(t, x, \int_0^t w(\tau,x)d\tau, \int_0^t \partial_2^p w(\tau, \alpha(t,x)x)d\tau\right) \tag{2.1}$$

for $(t,x) \in A(a,R)$. If, conversely, the integral equation (2.1) has a solution $w : A(a,R) \to \mathbf{C}$, then

$$u(t,x) := \int_0^t w(\tau,x)d\tau$$

becomes a solution in $A(a,R)$ of the Cauchy problem (1.5)-(1.2).

So we want now to solve the integral equation (2.1). For this purpose we take an upper bound M of $|f(t,x,u,v)|$ in $B(T,R,S)$. We need the following lemma.

Lemma 2.1. *If (t,x,u_1,v_1) and (t,x,u_2,v_2) are in $B(T,R,S/2)$, then the inequality*

$$\begin{aligned}&|f(t,x,u_1,v_1) - f(t,x,u_2,v_2)| \\ &\le \frac{4M}{S}\{|u_1-u_2| + |v_1-v_2|\}\end{aligned}$$

holds.

Proof . Easy by Cauchy's integral formula. □

Next let a be a positive constant. We denote by $\mathcal{C}(a)$ the set of all bounded continuous functions from $A(a,R)$ into $\mathbf{C}$ and write

$$\begin{aligned}\mathcal{D}(a) &= \{u \in \mathcal{C}(a)\,;\, u(t,x) \text{ is analytic in } x\}, \\ \mathcal{E}(a) &= \{u \in \mathcal{D}(a)\,;\, |u(t,x)| \le M \\ &\qquad \text{for all } (t,x) \in A(a,R)\}.\end{aligned}$$

$\mathcal{D}(a)$ becomes a Banach space by defining the norm of its element u by

$$\|u\| = \sup_{(t,x)\in A(a,R)} |u(t,x)|.$$

$\mathcal{E}(a)$ is a closed subset of $\mathcal{D}(a)$. Let us prepare another lemma.

Lemma 2.2. *If u is in $\mathcal{E}(a)$, then $\partial_2^p u(t,x)$ is continuous in (t,x), analytic in x and satisfies the inequality*

$$|\partial_2^p u(t,x)| \le \frac{p!M}{(R-|x|)^p}$$

for all $(t,x) \in A(a,R)$.

Proof . Easy by Cauchy's integral formula. □

Now let

$$a = \min\left\{T, \frac{S}{2M}, \frac{SR^p(1-m)^p}{p!2M}\right\} \tag{2.2}$$

and take an element w of the set $\mathcal{E}(a)$ arbitrarily. Then we have

$$\left|\int_0^t w(\tau,x)d\tau\right| \le |t|M \le aM \le S/2, \tag{2.3}$$

$$\begin{aligned}&\left|\int_0^t \partial_2^p w(\tau,\alpha(t,x)x)d\tau\right| \\ &\le \frac{p!M|t|}{(R-|\alpha(t,x)x|)^p} \le \frac{p!Ma}{R^p(1-m)^p} \le S/2.\end{aligned} \tag{2.4}$$

Therefore we see that, if a is defined by (2.2) and $w \in \mathcal{E}(a)$, then an element $\tilde{w} \in \mathcal{E}(a)$ is defined by

$$\tilde{w}(t,x) = f\left(t, x, \int_0^t w(\tau,x)d\tau, \int_0^t \partial_2^p w(\tau,\alpha(t,x)x)d\tau\right).$$

We denote the mapping $w \mapsto \tilde{w}$ by Φ. Let us check if the map $\Phi : \mathcal{E}(a) \to \mathcal{E}(a)$ is a contraction. Suppose two elements w_1 and w_2 of $\mathcal{E}(a)$ are given. Then we have, like (2.3) and (2.4),

$$\begin{aligned}&\left|\int_0^t w_1(\tau,x)d\tau - \int_0^t w_2(\tau,x)d\tau\right| \\ &\qquad \le a\|w_1 - w_2\|, \\ &\left|\int_0^t \partial_2^p w_1(\tau,\alpha(t,x)x)d\tau - \int_0^t \partial_2^p w_2(\tau,\alpha(t,x)x)d\tau\right| \\ &\qquad \le \frac{p!a\|w_1-w_2\|}{R^p(1-m)^p}.\end{aligned}$$

Therefore, by Lemma 2.1, we have

$$\begin{aligned}&\|\Phi(w_1) - \Phi(w_2)\| \\ &\le a\frac{4M}{S}\frac{R^p(1-m)^p + p!}{R^p(1-m)^p}\|w_1 - w_2\|.\end{aligned}$$

It follows that, if a satisfies the condition

$$\begin{aligned}a \le \min\Bigg\{&T, \frac{S}{2M}, \frac{SR^p(1-m)^p}{p!2M}, \\ &\frac{S}{8M}\frac{R^p(1-m)^p}{p! + R^p(1-m)^p}\Bigg\},\end{aligned} \tag{2.5}$$

then the inequality

$$\|\Phi(w_1) - \Phi(w_2)\| \le 2^{-1}\|w_1 - w_2\|$$

holds. This means that the map $\Phi : \mathcal{E}(a) \to \mathcal{E}(a)$ is a contraction, if a satisfies (2.5).

Therefore, there is a unique element $w \in \mathcal{E}(a)$ that satisfies the equality $\Phi(w) = w$. This means that the Cauchy problem (1.5)-(1.2) has a unique solution $u : A(a,R) \to \mathbf{C}$ with the properties mentioned in the theorem. $\square$

3. Shrinking in the time variable. The purpose of this section is to solve the Cauchy problem for the equation (1.6). Our result will be stated in the Theorem 3.1 below. In the theorem we use the following notation. If T and R are positive constants and p is a positive integer, then we write

$$\Omega(T,R,p) = \{(t,x) \in \mathbf{R} \times \mathbf{C}\,; \\ |t| < T,\ R - |x| - |t|^{1/(p+1)} > 0\}.$$

Theorem 3.1. *Let T, R, S and m be positive constant. Assume that $m<1$. Let p be an integer ≥ 1. In the differential equation* (1.6) *assume that*

(i) *$f(t,x,u,v)$ satisfies the same conditions as in* Theorem 2.1;

(ii) *$\alpha(t)$ is a real valued continuous function of $t \in [-T,T]$ satisfying the inequality $0 < \alpha(t) \leq m$.*

Then there is a positive constant a such that the Cauchy problem (1.6)-(1.2) *has a unique solution $u : \Omega(a,R,p) \to \mathbf{C}$ such that $u(t,x)$, $\partial_1 u(t,x)$ and $\partial_2^p u(t,x)$ are continuous in (t,x) and analytic in x.*

Proof . Assume that there exists a positive number $a \leq T$ such that the Cauchy problem (1.6)-(1.2) has a solution $u : \Omega(a,R,p) \to \mathbf{C}$. Then

$$w(t,x) := \partial_1 u(t,x)$$

satisfies the integral equation

$$w(t,x) = f\left(t,x,\int_0^t w(\tau,x)d\tau, \int_0^{\alpha(t)t} \partial_2^p w(\tau,x)d\tau\right) \tag{3.1}$$

in the region $\Omega(a,R,p)$. If, conversely, the integral equation (3.1) has a solution $w(t,x)$ in the region $\Omega(a,R,p)$, then

$$u(t,x) := \int_0^t w(\tau,x)d\tau$$

becomes a solution of the Cauchy problem (1.6)-(1.2).

So we want now to solve the integral equation (3.1). For this purpose we take an upper bound M of $|f(t,x,u,v)|$ in $B(T,R,S)$. Let a be a positive constant. We denote by $\mathcal{C}(a)$ the set of all bounded continuous functions from $\Omega(a,R,p)$ into $\mathbf{C}$ and write

$$\mathcal{D}(a) = \{u \in \mathcal{C}(a)\,;\, u(t,x) \text{ is analytic in } x\},$$
$$\mathcal{E}(a) = \{u \in \mathcal{D}(a)\,; \\ |u(t,x)| \leq M \text{ for all } (t,x) \in \Omega(a,R,p)\}.$$

$\mathcal{D}(a)$ becomes a Banach space by defining the norm of its element u by

$$\|u\| = \sup_{(t,x)\in\Omega(a,R,p)} |u(t,x)|.$$

$\mathcal{E}(a)$ is a closed subset of $\mathcal{D}(a)$. We need the following lemma.

Lemma 3.1. *If u is in $\mathcal{E}(a)$, then $\partial_2^p u(t,x)$ is continuous in (t,x), analytic in x and satisfies the inequality*

$$|\partial_2^p u(t,x)| \leq \frac{p!M}{\left(R - |x| - |t|^{1/(p+1)}\right)^p}$$

for all $(t,x) \in \Omega(a,R,p)$.

Proof . Easy by Cauchy's integral formula. $\square$

Corollary. *If u is in $\mathcal{E}(a)$ and (t,x) is in $\Omega(a,R,p)$, then*

$$|\partial_2^p u(mt,x)| \leq \frac{p!M}{\tilde{m}|t|^{p/(p+1)}},$$

where $\tilde{m} = \left(1 - m^{1/(p+1)}\right)^p$.

Proof . If $(t,x) \in \Omega(a,R,p)$, then $R - |x| - |t|^{1/(p+1)} > 0$ and

$$\begin{aligned}&R - |x| - |mt|^{1/(p+1)} \\ &= R - |x| - |t|^{1/(p+1)} + |t|^{1/(p+1)} - |mt|^{1/(p+1)} \\ &> (1 - m^{1/(p+1)})|t|^{1/(p+1)} = \tilde{m}^{1/p}|t|^{1/(p+1)}.\end{aligned}$$

Therefore we have, by Lemma 3.1,

$$\begin{aligned}&|\partial_2^p u(mt,x)| \\ &\leq \frac{p!M}{\left(R - |x| - |mt|^{1/(p+1)}\right)^p} \leq \frac{p!M}{\tilde{m}|t|^{p/(p+1)}}. \quad \square\end{aligned}$$

Now choose an $a > 0$ such that

$$a \leq \min\left\{T, \frac{S}{2M}, \left(\frac{S\tilde{m}}{(p+1)!2mM}\right)^{p+1}\right\} \tag{3.2}$$

and take an element w of the set $\mathcal{E}(a)$ arbitrarily. Then we have

$$\left|\int_0^t w(\tau,x)d\tau\right| \leq |t|M \leq aM \leq S/2. \tag{3.3}$$

Further we have, assuming that $t \geq 0$,

$$\begin{aligned}&\left|\int_0^{\alpha(t)t} \partial_2^p w(\tau,x)d\tau\right| \leq \int_0^{mt} |\partial_2^p w(\tau,x)|d\tau \\ &= m\int_0^t |\partial_2^p w(ms,x)|ds \leq m\int_0^t \frac{p!M}{\tilde{m}s^{p/(p+1)}}ds \\ &= \frac{p!mM}{\tilde{m}}(p+1)t^{1/(p+1)} \leq \frac{(p+1)!mM}{\tilde{m}}a^{1/(p+1)} \leq \frac{S}{2}.\end{aligned}$$

For $t \le 0$, too, we can discuss similarly. In conclusion we have the inequality

$$\left|\int_0^{\alpha(t)t} \partial_2^p w(\tau,x)d\tau\right| \le \frac{(p+1)!mM}{\tilde{m}} a^{1/(p+1)} \le \frac{S}{2} \tag{3.4}$$

that holds for all $(t,x) \in \Omega(a,R,p)$.

Therefore we see that, if a is defined by (3.2) and $w \in \mathcal{E}(a)$, then an element $\tilde{w}$ in $\mathcal{E}(a)$ is defined by

$$\tilde{w}(t,x) = f\left(t, x, \int_0^t w(\tau,x)d\tau, \int_0^{\alpha(t)t} \partial_2^p w(\tau,x)d\tau\right).$$

We denote the mapping $w \mapsto \tilde{w}$ by Φ. Let us check if the map $\Phi : \mathcal{E}(a) \to \mathcal{E}(a)$ is a contraction. Suppose two elements w_1 and w_2 of $\mathcal{E}(a)$ are given. Then we have, like (3.3) and (3.4),

$$\left|\int_0^t w_1(\tau,x)d\tau - \int_0^t w_2(\tau,x)d\tau\right| \le a\|w_1 - w_2\|$$

and

$$\left|\int_0^{\alpha(t)t} \partial_2^p w_1(\tau,x)d\tau - \int_0^{\alpha(t)t} \partial_2^p w_2(\tau,x)d\tau\right| \le m\tilde{m}^{-1}(p+1)!a^{1/(p+1)}\|w_1 - w_2\|.$$

Therefore, by Lemma 2.1, we have

$$|\Phi(w_1)(t,x) - \Phi(w_2)(t,x)| \le \frac{4M}{S}\left\{a + m\tilde{m}^{-1}(p+1)!a^{1/(p+1)}\right\}\|w_1 - w_2\|$$

and

$$\|\Phi(w_1) - \Phi(w_2)\| \le \frac{4M}{S}\left\{a + m\tilde{m}^{-1}(p+1)!a^{1/(p+1)}\right\}\|w_1 - w_2\|.$$

It follows that, if $a > 0$ satisfies the condition

$$a \le \min\left\{T, \frac{S}{2M}, \left(\frac{S\tilde{m}}{(p+1)!2mM}\right)^{p+1}, \tilde{a}\right\}, \tag{3.5}$$

where $\tilde{a}$ is the unique positive root of the equation

$$\frac{4M}{S}\{\tilde{a} + m\tilde{m}^{-1}(p+1)!\tilde{a}^{1/(p+1)}\} = \frac{1}{2},$$

then the inequality

$$\|\Phi(w_1) - \Phi(w_2)\| \le 2^{-1}\|w_1 - w_2\|$$

holds. This means that the map $\Phi : \mathcal{E}(a) \to \mathcal{E}(a)$ is a contraction, if $a > 0$ satisfies (3.5).

Therefore, there is a unique element $w \in \mathcal{E}(a)$ that satisfies the equality $\Phi(w) = w$. This means that the Cauchy problem (1.6)-(1.2) has a unique solution $u : \Omega(a,R,p) \to \mathbf{C}$ with the properties mentioned in the theorem. □

4. Remarks.

(i) It is easy to replace the equation (1.5) by more general ones such as

$$\partial_1 u(t,x) = f(t,x,u(t,x), \partial_2 u(t,\alpha_1(t,x)x), \cdots, \partial_2^p u(t,\alpha_p(t,x)x)). \tag{4.1}$$

(ii) It is also easy to consider the case where x and $u(t,x)$ in (1.5) or (4.1) are multi-dimensional.

(iii) The reason why $\alpha(t)$ in the differential equation (1.6) is cannot be replaced by $\alpha(t,x)$ is that we do not assume analyticity in t of $f(t,x,u,v)$. If we assume that $f(t,x,u,v)$ is analytic in (t,x,u,v), then we can replace $\alpha(t)$ by $\alpha(t,x)$ which is analytic in (t,x).

(iv) The story for the equation (1.6), too, can be generalized along the same lines as in the remarks (i) and (ii) above.

(v) The author has a plan to publish elsewhere the above mentioned generalizations of the stories for the equations (1.5) and (1.6).

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